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ELECTRON-STIMULATED ION OSCILLATIONS

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Abstract

Certain ion oscillations that are sometimes excited in microwave tubes were investigated on a theoretical basis. A study of some general properties of propagation in plasma-loaded waveguides was made; the plasma consisted of both stationary ions and an electron beam. The effect of an axial magnetic field was included in the analysis.

Specifically, the propagation in waveguides that were loaded merely with a stationary ion plasma was studied. It was found that these waveguides are propagating structures at frequencies that are very low, as compared with the empty waveguide cutoff frequency. A short-circuited section of an ion-loaded waveguide, which was considered as a resonant cavity, was found to have resonant frequencies of the order of the ion plasma frequency and ion cyclotron frequency (a few megacycles per second for positive gas ions such as hydrogen). The energy transfer from a beam traversing an oscillating ion-loaded cavity to the fields of the cavity was calculated, and a criterion for oscillation was established.

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Chapter I
INTRODUCTION AND GENERAL RELATIONS

I-1 Introduction

In a microwave tube, even under hard vacuum conditions, there is a considerable number of gas molecules present, some of which become ionized by the electron beam. Depending upon the geometry and dc potentials of the various tube elements, the ions which are positively charged can become entrapped in the tube in certain regions termed "ion traps". A drift tube held at a dc potential slightly lower than both the accelerating anode and collector potentials constitutes an ion trap. The ions formed in a trap accumulate until some sort of equillibrium is reached. Because of an abundance of gas molecules and a scarcity of drainage paths for the trapped positive ions, the ions can accumulate to such an extent that the charge in the beam becomes neutralized, or perhaps even over-neutralized. Ions trapped in a potential trough have in fact been used for beam focusing by neutralizing the beam.[1]

Experimental observations show that the electron beam sometimes excites oscillations of the trapped ions. Several types of oscillations are observed under different conditions; Cutler[2] and Moreno[3] have nicely summarized these observations. The most prominent phenomena observed are sustained sinusoidal oscillations of the ion-electron beam plasma. These sinusoidal oscillations are often strong enough to noticeably perturb the desired output of the tube, the pertubations appearing as amplitude modulations of the rf output. The modulation frequencies, the frequencies of the plasma oscillations, are roughly in the neighborhood of the ion plasma frequency ω_{pi} and the ion cyclotron frequency ω_{ci} (of the order of a few Mc/sec). This thesis is concerned with predicting and analyzing these sustained sinusoidal oscillations.

The general system considered has an electron beam which drifts through a drift tube (or waveguide) containing trapped (non-drifting) positive ions, a drift region being characterized by the absence of dc electric fields which would accelerate the beam. The problem is to obtain solutions for electromagnetic waves in the plasma, to combine these solutions to match boundary conditions, and to find the mechanism that sustains oscillations.

I-2 General Assumptions and Equations

In the following analysis, the electron beam, in the absence of an ac excitation, is assumed to be drifting with a constant uniform velocity $\bar{v}_0 = \bar{i}_z v_0$ ($v_0 \ll c$) and to have a uniform volume charge density $-\rho_0$. The trapped positive ions, in the absence of an ac excitation, are assumed to have a uniform volume charge density $+\rho_0$ and no drift velocity. The electron beam drifts through the positive ion cloud and it is assumed (again in the absence of an ac excitation) both the beam and the ion cloud occupy the same space so that there is no net dc space charge - - hence there is no dispersion of the beam or ions.

Maxwell's equations and the force equation are used to represent the system. In Maxwell's equations, the currents due to the motion of the particles are included. Thermal motion and collisions are neglected and the motion of the particles is assumed to be due to the action of the electromagnetic fields alone. The effect of a uniform axial magnetic focusing field $\bar{B}_0 = \bar{i}_z B_0$ is also considered in the analysis.

To linearize the equations, a small signal analysis is employed. Each variable is expressed by a dc quantity plus an ac pertubation. The dc quantity is the value of the variable in the absence of an ac excitation. The ac pertubation, referred to as the "small signal quantity", is assumed to be very much smaller in amplitude than the dc quantity - - this is the "small signal assumption". In a small signal analysis the products of small signal quantities are neglected since they are of second order.

It is assumed that all ac quantities have a z - dependence of $e^{-j\beta_z z}$ where β_z is the propagation constant. With the assumed z - dependence and the small signal assumption, the following set of linear equations, which describe the ion-electron beam plasma are obtained after Maxwell's equations and the force equations are properly manipulated, time dependence $e^{j\omega t}$ being assumed.

$$\nabla^2\bar{E} + k^2\bar{E} = j\omega\mu\bar{J} + \frac{\nabla\rho}{\epsilon_0} \tag{I-2.1}$$

$$\nabla^2\bar{H} + k^2\bar{H} = -\nabla x\bar{J} \tag{I-2.2}$$

$$\rho = \rho_0\left[\frac{\nabla\cdot\bar{v}_e}{j\omega(1-\frac{\beta_z}{\beta_e})} - \frac{\nabla\cdot\bar{v}_i}{j\omega}\right] \tag{I-2.3}$$

$$\bar{J} = \rho_0\left[\bar{v}_i - \bar{v}_e + \frac{\nabla\cdot\bar{v}_e}{j\omega(1-\frac{\beta_z}{\beta_e})}\bar{v}_0\right] \tag{I-2.4}$$

$$\frac{-j\omega m_e}{e}(1-\frac{\beta_z}{\beta_e})\bar{v}_e = \bar{E} + \bar{v}_e \times \bar{B}_0 + \mu_0\bar{v}_0 \times \bar{H} \tag{I-2.5}$$

$$\frac{j\omega m_i}{q}\bar{v}_i = \bar{E} + \bar{v}_i \times \bar{B}_0 \tag{I-2.6}$$

The variables $\bar{E}$, $\bar{H}$, ρ, $\bar{J}$, $\bar{v}_e$, and $\bar{v}_i$, are complex small signal quantities that are functions of the space coordinates. These equations, which are applied in the ensuing chapters, are derived in detail in Appendix A.

I-3 Boundary Conditions

In the presence of an ac excitation, the boundary of the ion-electron beam plasma is rippled. By the method introduced by Hahn[4], this ripple can be approximated by a sheet of ac surface charge and surface current at the same boundary the plasma has in the absence of excitation, the dc boundary so to speak.

For example, for the case of a circular cylindrical plasma of dc radius b, the ripple at the plasma boundary is replaced at r= b by an ac charge sheet of surface charge density

$$\sigma = \rho_0 \left[\frac{v_{ir}}{j\omega} - \frac{v_{er}}{j\omega(1- \frac{\beta_z}{\beta_e})} \right]_{r = b} \qquad \text{(I-3.1)}$$

and an ac current sheet of linear current density

$$\bar{K} = \bar{i}_z K_z = \bar{i}_z \left. \frac{j\rho_o v_{er}}{\beta e - \beta_z} \right|_{r = b} \qquad \text{(I-3.2)}$$

These expressions are derived in Appendix B.

The field solutions in the plasma and in the "vacuum"* are matched at the ac charge and current sheets on the dc boundary. At this surface, we apply the well known set of boundary conditions

$$\bar{n} \times (\bar{E}^{II} - \bar{E}^{I}) = 0 \qquad \text{(I-3.3)}$$

$$\bar{n} \cdot (\bar{E}^{II} - \bar{E}^{I}) = \sigma / \epsilon_0 \qquad \text{(I-3.4)}$$

$$\bar{n} \times (\bar{H}^{II} - \bar{H}^{I}) = \bar{K} \qquad \text{(I-3.5)}$$

$$\bar{n} \cdot (\bar{H}^{II} - \bar{H}^{I}) = 0 \qquad \text{(I-3.6)}$$

where $\bar{n}$ is the unit vector normal to the boundary surface and pointing from the plasma to the vacuum. The superscripts I and II refer to the plasma and vacuum quantities, respectively. The field quantities in the above boundary conditions are computed at the surface of the plasma.

It is not necessary to apply all four of these boundary conditions. If the relation (I-3.4) is given, then the continuity equation at the surface

* Here, "vacuum" means un-ionized gas. This region is referred to as a vacuum since its electrical properties are considered the same as a true vacuum.

$$j\omega\sigma + \nabla\cdot\bar{K} - n\cdot\bar{J}^{I} = 0 \qquad \text{(I-3.7)}$$

reduces to the relation (I-3.5); the reverse is also true, given (I-3.5),(I-3.7) reduces to (I-3.4); hence these two boundary conditions are not independent. Similarly, if either relation (I-3.3) or (I-3.6) is given, then the other can be derived from Maxwell's equations in the two media; hence (I-3.3) and (I-3.6) are not independent. Therefore, only two of these boundary conditions, either (I-3.3) and (I-3.6) or (I-3.4) and (I-3.5), need be applied.

Metal walls are assumed to be perfectly conducting. Grids will be assumed to look like smooth perfectly conducting walls to $\bar{E}$ and $\bar{H}$ fields but transparent to electrons.

Chapter II

FIELD SOLUTIONS IN RECTANGULAR COORDINATES

II-I Introductory Remarks

The set of equations (I-2.1) to (I-2.6) is difficult to solve, even though many simplifying assumptions are made in their derivation. However, if we look for solutions in a rectangular coordinate system, the differential equations reduce to linear algebraic equations which are more readily soluble.

If it is assumed that each of the parameters has the spatial dependence $e^{-j\bar{\beta}\cdot\bar{r}}$ in which

$$\bar{\beta} = \bar{i}_x\beta_x + \bar{i}_y\beta_y = \bar{i}_z\beta_z$$

and

$$\bar{r} = \bar{i}_x x + \bar{i}_y y + \bar{i}_z z \ ,$$

then the vector differential operator

$$= \bar{i}_x \frac{\partial}{\partial x} + \bar{i}_y \frac{\partial}{\partial y} + \bar{i}_z \frac{\partial}{\partial z}$$

is replaced by the vector constant $-j\bar{\beta}$. The vector calculus operations become transformed to vector algebra operations as follows:

Gradient	∇f	$\longrightarrow$	$-j\bar{\beta}f$
Scalar Laplacian	$\nabla^2 f$	$\longrightarrow$	$-\beta^2 f$
Curl	$\nabla x \bar{F}$	$\longrightarrow$	$-j\bar{\beta} \times \bar{F}$
Divergence	$\nabla \cdot \bar{F}$	$\longrightarrow$	$-j\bar{\beta}\cdot\bar{F}$
Vector Laplacian	$\nabla^2 \bar{F}$	$\longrightarrow$	$-\beta^2 \bar{F}$

where f is a scalar function of $\bar{r}$, $\bar{F}$ is a vector function of $\bar{r}$ (the $\bar{r}$ dependence being of the form $e^{-j\bar{\beta}\cdot\bar{r}}$), and

$$\beta^2 = \bar{\beta}\cdot\bar{\beta} = \beta_x^2 + \beta_y^2 + \beta_z^2$$

In general, the propagation vector $\bar{\beta}$ is complex in both magnitude and direction and we have a non-uniform plane wave; for the special case when $\bar{\beta}$ can be expressed as the product of a complex scalar and a real vector, we have a uniform plane wave. By the superposition of plane waves of different propagation vectors, wave configurations which fit finite boundaries can be synthesized.

Utilization of the above transformations convert the differential equations to algebraic equations. From the algebraic equations, the determinantal equation, the expression that relates the components of $\bar{\beta}$ to the physical constants (ω, ρ_o, v_o, B_o, e, q, m_e, m_i), can be found. Also, the relationships among the various field components of the exponential waves can be determined in terms of the components of $\bar{\beta}$ and physical constants; the transverse components of $\bar{\beta}$, β_x and β_y, are determined by transverse boundary conditions. As will be seen in the following sections of this chapter, determination of these relations requires a frustrating amount of mathamatical manipulations even though the calculus operations are replaced by algebraic operations.

II-2 General Solutions in the Ion - Electron Beam Plasma

For solutions of the form $e^{-j\bar{\beta}\cdot\bar{r}}$, Eqs. (I-2.1) to (I-2.4) become

$$(-\beta^2 + k^2)\bar{E} = j\omega\mu\bar{J} - j\bar{\beta}\rho/\epsilon_o \tag{II-2.1}$$

$$(-\beta^2 + k^2)\bar{H} = j\bar{\beta}x\bar{J} \tag{II-2.2}$$

$$\rho = \rho_o\left[\frac{-\bar{\beta}\cdot\bar{v}_e}{\omega(1-\frac{\beta_z}{\beta_e})} + \frac{\bar{\beta}\cdot\bar{v}_i}{\omega}\right] \tag{II-2.3}$$

$$\bar{J} = \rho_o\left[\bar{v}_i - \bar{v}_e - \frac{\bar{\beta}\cdot\bar{v}_e}{\omega(1-\frac{\beta_z}{\beta_e})}\,\bar{v}_o\right] \tag{II-2.4}$$

$$\frac{-j\omega m_e}{e}\,(1-\frac{\beta_z}{\beta_e})\,\bar{v}_e = \bar{E} + \bar{v}_e \times \bar{B}_o + \mu_o\bar{v}_o \times \bar{H} \tag{II-2.5}$$

$$\frac{j\omega m_i}{q}\,\bar{v}_i = \bar{E} + \bar{v}_i \times \bar{B}_o \tag{II-2.6}$$

Note that Eqs. (I-2.5) and (I-2.6) are not affected by the transformations and are just rewritten as (II-2.5) and (II-2.6) for convenience.

The above set of equations represents sixteen linear scalar equations within sixteen scalar unknowns, since each of the vector equations represents three scalar equations. These equations can be condensed into two vector equations, representing six scalar equations, by the elimination of some of the variables. When Eqs. (II-2.3) and (II-2.4) are substituted into Eqs. (II-2.1) and (II-2.2), $\bar{E}$ and $\bar{H}$ can be expressed in terms of $\bar{v}_e$ and $\bar{v}_i$ with the result that

$$\bar{E} = \frac{j\omega\mu\rho_o}{-\beta^2+k^2}\left\{\bar{v}_i-\bar{v}_e-\bar{i}_z\frac{\bar{\beta}\cdot\bar{v}_e}{\beta_e(1-\frac{\beta_z}{\beta_e})}+\frac{\bar{\beta}}{k^2}\left[\frac{\bar{\beta}\cdot\bar{v}_e}{1-\frac{\beta_z}{\beta_e}}-\bar{\beta}\cdot\bar{v}_i\right]\right\} \qquad \text{(II-2.7)}$$

and

$$\bar{H} = \frac{j\rho_o}{-\beta^2+k^2}\,\bar{\beta}\times\left[\bar{v}_i-\bar{v}_e-\bar{i}_z\frac{\bar{\beta}\cdot\bar{v}_e}{\beta_e(1-\frac{\beta_z}{\beta_e})}\right] \qquad \text{(II-2.8)}$$

Substitution of Eqs. (II-2.7) and (II-2.8) into Eqs. (II-2.5) and (II-2.6) yields

$$-(1-\frac{\beta_z}{\beta_e})\,\bar{v}_e = \frac{\omega_{pe}^2}{\omega^2}\,\frac{k^2}{-\beta^2+k^2}\left[\bar{v}_i-\bar{v}_e-\bar{i}_z\frac{\bar{\beta}\cdot\bar{v}_e}{\beta_e(1-\frac{\beta_z}{\beta_e})}\right]$$

$$+\frac{\omega_{pe}^2}{\omega^2}\,\frac{\bar{\beta}}{-\beta^2+k^2}\left[\frac{\bar{\beta}\cdot\bar{v}_e}{1-\frac{\beta_z}{\beta_e}}-\bar{\beta}\cdot\bar{v}_i\right]-j\frac{\omega_{ce}}{\omega}\,\bar{v}_e\times\bar{i}_z$$

$$+\frac{\omega_{pe}^2}{\omega^2}\,\frac{k^2}{-\beta^2+k^2}\,\bar{i}_z\times\frac{\bar{\beta}}{\beta_e}\times\left[\bar{v}_i-\bar{v}_e-\bar{i}_z\frac{\bar{\beta}\cdot\bar{v}_e}{\beta_e(1-\frac{\beta_z}{\beta_e})}\right] \qquad \text{(II-2.9)}$$

and

$$\bar{v}_i = \frac{\omega_{pi}^2}{\omega^2}\,\frac{k^2}{-\beta^2+k^2}\left[\bar{v}_i-\bar{v}_e-\bar{i}_z\frac{\bar{\beta}\cdot\bar{v}_e}{\beta_e(1-\frac{\beta_z}{\beta_e})}\right]$$

$$+\frac{\omega_{pi}^2}{\omega^2}\,\frac{\bar{\beta}}{-\beta^2+k^2}\left[\frac{\bar{\beta}\cdot\bar{v}_e}{1-\frac{\beta_z}{\beta_e}}-\bar{\beta}\cdot\bar{v}_i\right]-j\frac{\omega_{ci}}{\omega}\,\bar{v}_i\times\bar{i}_z \qquad \text{(II-2.10)}$$

where

$$\omega_{pe}^2 = \frac{e\rho_o}{\epsilon_o m_e},\ \omega_{pi}^2 = \frac{q\rho_o}{\epsilon_o m_e},\ \omega_{ce} = \frac{eB_o}{m_e},\ \text{and}\ \ \omega_{ci} = \frac{qB_o}{m_i}.$$

ω_{pe} and ω_{pi} are the electron and ion plasma frequencies, respectively; and ω_{ce} and ω_{ci} are the electron and ion cyclotron frequencies, respectively.

When the vector operations in Eqs.(II-2.9) and (II-2.10) are performed, the resulting set of scalar equations can be expressed as a matrix multiplication

$$\left[C_v\right] \left.v\right] = \left.0\right] \qquad \text{(II-2.11)}$$

The column matrix of small signal velocities is

$$\left.v\right] = \begin{bmatrix} v_{ex} \\ v_{ey} \\ v_{ez} \\ v_{ix} \\ v_{iy} \\ v_{iz} \end{bmatrix}$$

The velocity coupling matrix C_v is displayed below in Eq. (II-2.11a).

$$
\left[\begin{array}{c:c:c:c:c:c}
-\beta^2+k^2+\frac{\omega_{pe}^2}{\omega^2}\left[-k^2+\frac{1-k^2/\beta_e^2}{1-\beta_z/\beta_e}\beta_x^2\right] & \frac{\omega_{pe}^2}{\omega^2}\frac{1-k^2/\beta_e^2}{(1-\beta_z/\beta_e)^2}\beta_x\beta_y-j\frac{\omega_{ce}}{\omega}\left(\frac{-\beta^2+k^2}{1-\beta_z/\beta_e}\right) & \frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{(1-\beta_z/\beta_e)^2}\beta_x & \frac{\omega_{pe}^2}{\omega^2}\left[k^2-\frac{\beta_x^2}{1-\beta_z/\beta_e}\right] & -\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_x\beta_y}{1-\beta_z/\beta_e} & -\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{1-\beta_z/\beta_e}\beta_x \\ \hdashline
\frac{\omega_{pe}^2}{\omega^2}\frac{1-k^2/\beta_e^2}{(1-\beta_z/\beta_e)^2}\beta_x\beta_y+j\frac{\omega_{ce}}{\omega}\left(\frac{-\beta^2+k^2}{1-\beta_z/\beta_e}\right) & -\beta^2+k^2+\frac{\omega_{pe}^2}{\omega^2}\left[-k^2+\frac{1-k^2/\beta_e^2}{1-\beta_z/\beta_e}\beta_y^2\right] & \frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{(1-\beta_z/\beta_e)^2}\beta_y & -\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_x\beta_y}{1-\beta_z/\beta_e} & \frac{\omega_{pe}^2}{\omega^2}\left[k^2-\frac{\beta_y^2}{1-\beta_z/\beta_e}\right] & -\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{1-\beta_z/\beta_e}\beta_y \\ \hdashline
\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{(1-\beta_z/\beta_e)^2}\beta_x & \frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{(1-\beta_z/\beta_e)^2}\beta_y & -\beta^2+k^2+\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z^2-k^2}{(1-\beta_z/\beta_e)^2} & -\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_x\beta_z}{1-\beta_z/\beta_e} & -\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_y\beta_z}{1-\beta_z/\beta_e} & -\frac{\omega_{pe}^2}{\omega^2}\frac{\beta_z^2-k^2}{1-\beta_z/\beta_e} \\ \hdashline
\frac{\omega_{pi}^2}{\omega^2}\left[k^2-\frac{\beta_x^2}{1-\beta_z/\beta_e}\right] & -\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_x\beta_y}{1-\beta_z/\beta_e} & -\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_x\beta_z}{1-\beta_z/\beta_e} & -\beta^2+k^2+\frac{\omega_{pi}^2}{\omega^2}(\beta_x^2-k^2) & \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_y+j\frac{\omega_{ci}}{\omega}(-\beta^2+k^2) & \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_z \\ \hdashline
-\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_x\beta_y}{1-\beta_z/\beta_e} & \frac{\omega_{pi}^2}{\omega^2}\left[k^2-\frac{\beta_y^2}{1-\beta_z/\beta_e}\right] & -\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_y\beta_z}{1-\beta_z/\beta_e} & \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_y-j\frac{\omega_{ci}}{\omega}(-\beta^2+k^2) & -\beta^2+k^2+\frac{\omega_{pi}^2}{\omega^2}(\beta_y^2-k^2) & \frac{\omega_{pi}^2}{\omega^2}\beta_y\beta_z \\ \hdashline
-\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{1-\beta_z/\beta_e}\beta_x & -\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_z-k^2/\beta_e}{1-\beta_z/\beta_e}\beta_y & -\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_z^2-k^2}{1-\beta_z/\beta_e} & \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_z & \frac{\omega_{pi}^2}{\omega^2}\beta_y\beta_z & -\beta^2+k^2+\frac{\omega_{pi}^2}{\omega^2}(\beta_z^2-k^2)
\end{array}\right] \qquad \text{(II-2.11 a)}
$$

The matrix equation represents a homogeneous set - - this means that the velocity coupling matrix C_v is singular, or the determinant of C_v is equal to zero. The relationship among the components of $\bar{\beta}$ is given by the determinantal equation

$$\left| C_v \right| = 0.$$

The expansion of this determinant for the general case is rather complicated. If the complete determinant were expanded, the significance of each of the physical parameters in the expression would be obscured by the complexity of the algebra. Likewise, if algebraic expressions were obtained for the relationships between the various field components, each as E_x/E_z, v_{ex}/v_{ey}, etc., the physical picture would be hidden by the complexity of the algebra. So further simplifications are necessary for obtaining clear and workable expressions.

Before further simplifications are considered, special cases of the determinant of C_v will be examined.

II-3 Infinite Parallel Plane Case

In many problems it is often helpful to study a very simplified version just to obtain an intuitive feeling for the more complicated problems. In problems dealing with field solutions, the simplified approach is to assume that there is no transverse dependence and that the solutions are functions of the coordinate in the direction of propagation alone, a one dimensional dependence. For instance, in the elementary space charge analysis of the klystron, it is assumed that there is a beam of infinite cross-section drifting in the z-direction between infinite parallel plane grids, the electromagnetic and velocity fields in the beam having only z-dependence.

As a simplified version of our problem, we will assume in this section that the ion-electron beam plasma is infinite in the x and y directions so that all the field quantities depend only upon z. This means that, in the rectangular coordinate system, $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$

or

$$\beta_x = \beta_y = 0 \qquad \text{(II-3.1)}$$

and

$$\bar{\beta} = \bar{i}_z \beta_z \qquad \text{(II-3.2)}$$

Substituting Eqs. (II-3.1) and (II-3.2) into Eq. (II-2.11) we obtain

$$\left[C_{v1}\right] \left.v\right] = \left.0\right] \qquad \text{(II-3.3)}$$

where C_{v1} is the one dimensional velocity coupling matrix given by

$$\left[C_{v1}\right] =$$

$$\begin{bmatrix}
-\beta_z^2 + k^2\left(1-\frac{\omega_{pe}^2}{\omega^2}\right) & -j\frac{\omega_{ce}}{\omega}\left(\frac{-\beta_z^2+k^2}{1-\beta_z/\beta_e}\right) & 0 & k^2\frac{\omega_{pe}^2}{\omega^2} & 0 & 0 \\
j\frac{\omega_{ce}}{\omega}\left(\frac{-\beta_z^2+k^2}{1-\beta_z/\beta_e}\right) & -\beta_z^2 + k^2\left(1-\frac{\omega_{pe}^2}{\omega^2}\right) & 0 & 0 & k^2\frac{\omega_{pe}^2}{\omega^2} & 0 \\
0 & 0 & \left(-\beta_z^2+k^2\right)\left[1 - \frac{\omega_{pe}^2}{\omega^2}\frac{1}{\left(1-\beta_z/\beta_e\right)^2}\right] & 0 & 0 & \frac{\omega_{pe}^2}{\omega^2}\left(\frac{-\beta_z^2+k^2}{1-\beta_z/\beta_e}\right) \\
k^2\frac{\omega_{pi}^2}{\omega^2} & 0 & 0 & -\beta_z^2 + k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right) & j\frac{\omega_{ci}}{\omega}\left(-\beta_z^2+k^2\right) & 0 \\
0 & k^2\frac{\omega_{pi}^2}{\omega^2} & 0 & -j\frac{\omega_{ci}}{\omega}\left(-\beta_z^2+k^2\right) & -\beta_z^2 + k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right) & 0 \\
0 & 0 & \frac{\omega_{pi}^2}{\omega^2}\left(\frac{-\beta_z^2+k^2}{1-\beta_z/\beta_e}\right) & 0 & 0 & \left(-\beta_z^2+k^2\right)\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)
\end{bmatrix}$$

(II-3.4)

In the matrix C_{v1} there are no elements which couple together longitudinal and transverse components of velocity. Hence the longitudinal and transverse components of velocity are mutually independent, and Eq. (II-3.3) can be split into two independent relations

$$\left[C_{v1}^{t}\right] v^{t}\Big] = 0\Big] \qquad \text{(II-3.5)}$$

and

$$C_{v1}^{z} \; v^{z} = 0 \qquad \text{(II-3.6)}$$

where the transverse velocity coupling matrix $C_{v1}^{t} =$

$$\begin{bmatrix} -\beta_z^2+k^2\left(1-\frac{\omega_{pe}^2}{\omega^2}\right) & -j\,\frac{\omega_{ce}}{\omega}\,\frac{(-\beta_z^2+k^2)}{1-\beta_z/\beta_e} & \frac{\omega_{pe}^2}{\omega^2}\,k^2 & 0 \\ j\,\frac{\omega_{ce}}{\omega}\,\frac{(\beta_z^2+k^2)}{1-\beta_z/\beta_e} & -\beta_z^2+k^2\left(1-\frac{\omega_{pe}^2}{\omega^2}\right) & 0 & \frac{\omega_{pe}^2}{\omega^2}\,k^2 \\ \frac{\omega_{pi}^2}{\omega^2}\,k^2 & 0 & -\beta_z^2+k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right) & j\frac{\omega_{ci}}{\omega}(-\beta_z^2+k^2) \\ 0 & \frac{\omega_{pi}^2}{\omega^2}\,k^2 & -j\,\frac{\omega_{ci}}{\omega}\,(-\beta_z^2+k^2) & -\beta_z^2+k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right) \end{bmatrix}$$

(II-3.7)

the longitudinal velocity coupling matrix $C_{v1}^{z} =$

$$\begin{bmatrix} (-\beta_z^2+k^2)\left[1-\frac{\omega_{pe}^2}{\omega^2}\,\frac{1}{(1-\frac{\beta_z}{\beta_e})^2}\right] & \frac{\omega_{pe}^2}{\omega^2}\,\frac{(-\beta_z^2+k^2)}{1-\frac{\beta_z}{\beta_e}} \\ \frac{\omega_{pi}^2}{\omega^2}\,\frac{-\beta_z^2+k^2}{1-\frac{\beta_z}{\beta_e}} & (-\beta_z^2+k^2)\left(1-\frac{\omega_{pi}^2}{\omega^2}\right) \end{bmatrix} \qquad \text{(II-3.8)}$$

and the transverse and longitudinal velocity matrices are

$$v^t] = \begin{bmatrix} v_{ex} \\ v_{ey} \\ v_{ix} \\ v_{iy} \end{bmatrix} \qquad v^z] = \begin{bmatrix} v_{ez} \\ v_{iz} \end{bmatrix} .$$

For pure transverse small signal velocities ($\bar{v}_e = \bar{v}_e{}^t$ and $\bar{v}_i = \bar{v}_i{}^t$) Eqs.(II-2.3) and (II-2.4) show that

$$\rho = 0 \quad \text{(no bunching of charge)}$$

and

$$\bar{J} = \rho_o \left[\bar{v}_i{}^t - \bar{v}_e{}^t \right] = \bar{J}^t$$

Hence Eqs.(II-2.1) and (II-2.2) become

$$(-\beta_z{}^2 + k^2)\, \bar{E}^t = j\omega\mu \bar{J}^t \tag{II-3.9}$$

and

$$(-\beta_z{}^2 + k^2)\, \bar{H}^t = j\bar{\beta} \times \bar{J}^t \tag{II-3.10}$$

from which $\bar{E}$ and $\bar{H}$ are also purely transverse. From Eq.(II-3.9)

$$\bar{J}^t + j\omega\epsilon_o \bar{E}^t = - \frac{\beta_z{}^2}{\omega\mu} \bar{E}^t$$

and Maxwell's equations, Eqs.(A-1) to (A-4), become

$$\bar{i}_z \times \bar{H}^t = - \frac{\beta_z}{\omega\mu} \bar{E}^t$$

$$\bar{i}_z \times \bar{E}^t = \frac{\omega\mu}{\beta_z} \bar{H}^t$$

$$\bar{i}_z \cdot \bar{E}^t = 0$$

$$\bar{i}_z \cdot \bar{H}^t = 0$$

Hence we see that the vectors $\bar{E}$, $\bar{H}$, and $\bar{\beta}$ are mutually perpendicular (Fig.II-1) and that a wave, in which only transverse velocities are excited, is an infinite parallel plane TEM wave propagating in the z direction. This wave is in general elliptically polarized.

For pure longitudinal small signal velocities ($\bar{v}_e = \bar{i}_z v_{ez}$ and $\bar{v}_i = \bar{i}_z v_{iz}$), Eqs.(II-2.3) shows that the current is also purely longitudinal ($\bar{J} = \bar{i}_z J_z$) and Eqs.(II-2.1) and (II-2.2) become

$$(-\beta_z^{\,2} + k^2)\,\bar{E} = \bar{i}_z\,(j\omega\mu J_z - j\beta_z \rho/\epsilon_o)$$

and

$$(-\beta_z^{\,2} + k^2)\,\bar{H} = j\bar{\beta} \times \bar{i}_z J_z = 0$$

showing that $\bar{E} = \bar{i}_z E_z$ and $\bar{H} = 0$. Then, for pure longitudinal velocities, we have an electromagnetic wave, that consists entirely of a longitudinal component of electric field which is related to the current through

$$J_z + j\omega\epsilon_o E_z = 0 \qquad \text{(II-3.11)}$$

from Eq.(A-1)

The determinantal equation for the one dimensional case is the expansion of the determinant of the matrix C_{v1} equated to zero. Because of the normal modes discovered above, the determinant of C_{v1} becomes

$$\left|C_{v1}\right| = \left|C_{v1}^{\,z}\right| \; \left|C_{v1}^{\,t}\right| = 0$$

where C_{v1}^{z} is the determinant for the longitudinal wave and C_{v1}^{t} is the determinant for the transverse wave.

The longitudinal and transverse waves correspond to the "electrostatic" and "electromagnetic" waves discussed in Spitzer's book[5]; the Hydromagnetic, or Alfvén, waves discussed therein do not apply here, since they arise when the magnetic field is transverse to the direction of propagation.

From Eq.(II-3.8), $\left|C_{v\,1}^{\,z}\right| = 0$ becomes

$$\beta_z = \beta_e \pm \frac{\beta_{pe}}{\sqrt{1 - \frac{\omega_{pi}^2}{\omega^2}}} \tag{II-3.12}$$

where $\beta_{pe} = \omega_{pe}/v_o$, and we have the dispersion relation for the longitudinal wave; this expression is the same as that derived by Pierce [6]. Equation (II-3.12) indicates that for $\omega > \omega_{pi}$, there are two possible propagating waves that can be excited, one with a phase velocity faster than v_o and the other with a phase velocity slower than v_o. For $\omega < \omega_{pi}$ there are two possible waves, each travelling with the positive phase velocity v_o, but one attenuates while the other grows as indicated by the complex β_z. At $\omega = \omega_{pi}$, the ion plasma frequency β_z becomes infinite.

The implicit relation for the propagation constant of the transverse wave is found from Eq.(II-3.7) by the expansion of the determinant $\left|C_{v\,1}^{\,t}\right| = 0$,

$$\left\{\left[-\beta_z^2+k^2\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)\right]^2 - \frac{\omega_{ce}^2}{\omega^2}\left(\frac{-\beta_z^2+k^2}{1-\beta_z/\beta_e}\right)^2\right\}\left\{\left[-\beta_z^2+k^2\left(1-\frac{\omega_{ci}^2}{\omega^2}\right)\right]^2 - \frac{\omega_{ci}^2}{\omega^2}\left(-\beta_z^2+k^2\right)^2\right\} - 2k^4\left\{\left[-\beta_z^2+k^2\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)\right]\left[-\beta_z^2 + k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)\right] - \frac{\omega_{ce}}{\omega}\,\frac{\omega_{ci}}{\omega}\,\frac{(-\beta_z^2+k^2)^2}{1-\beta_z/\beta_e}\right\} + \frac{\omega_{pi}^4}{c^4}\,\frac{\omega_{pe}^4}{c^4} = 0. \tag{II-3.13}$$

It is rather hopeless to try to extract β_z as an explicit function of ω from Eq.(II-3.13) without further simplifications. If we consider a system consisting of simply ions alone, then Eq.(II-3.13) is greatly simplified and we arrive at the expression

$$\beta_z^2 = k^2 \left[1 - \frac{\omega_{pi}^2/\omega^2}{1 \pm \omega_{ci}/\omega} \right] ; \qquad \text{(II-3.14)}$$

this result is in agreementwith Spitzer[7]. For $\omega_{ci} = 0$, we have the familiar "ionospheric waves" - the plasma behaves as a medium whose permitivity is

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right)$$

For $\omega_{ci} = 0$, the two ionspheric waves split into four new waves as indicated by the $\pm$ sign. The case of ions alone will be further discussed in this chapter and investigated in the following chapters.

II-4 Infinite Magnetic Field

If we let B_0 become infinite, then from the force equations, Eqs.(II-2.5) and (II-2.6), we see that the transverse velocity components vanish, leaving only longitudinal velocities. Hence, the sixth order determinant, Eq.(II-2.11a), reduces to one of second order,

$$\begin{vmatrix} -\beta^2+k^2+\dfrac{\omega_{pe}^2}{\omega^2}\,\dfrac{\beta_z^2-k^2}{\left(1-\frac{\beta_z}{\beta_e}\right)^2} & -\dfrac{\omega_{pe}^2}{\omega^2}\,\dfrac{\beta_z^2-k^2}{1-\frac{\beta_z}{\beta_e}} \\ -\dfrac{\omega_{pi}^2}{\omega^2}\,\dfrac{\beta_z^2-k^2}{1-\frac{\beta_z}{\beta_e}} & -\beta^2+k^2+\dfrac{\omega_{pi}^2}{\omega^2}\left(\beta_z^2-k^2\right) \end{vmatrix} = 0 \qquad \text{(II-4.1)}$$

This determinant when expanded yields the determinantal equation

$$\left\{-\beta_T^2 + (-\beta_z^2+k^2)\left[1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\beta_{pe}^2}{(\beta_e-\beta_z)^2}\right]\right\}\left[-\beta_T^2-\beta_z^2+k^2\right] = 0 \quad \text{(II-4.2)}$$

where $\beta_T^2 = \beta_x^2 + \beta_y^2$ and $\beta_{pe} = \omega_{pe}/v_o$. The quantity β_T^2 is determined by transverse boundary conditions

At this point we will examine how plane waves are added to give the field solutions. To synthesize fields that fit the boundary conditions from plane wave solutions, we must consider all the waves which have the same β_z at the same frequency; in other words, we must look at waves which have the same dispersion relation. It is clear from Eq.(II-4.2) that if each of the individual plane waves gives the same β_T^2, the dispersion relations will be the same. Hence the electric field becomes

$$\bar{E} = e^{-j\beta_z z} \sum_{k=1}^{4} \bar{A}_k \; e^{-j\bar{\beta}_{Tk}\cdot\bar{r}_T} \quad \text{(II-4.3)}$$

where

$$\bar{r}_T = \bar{i}_x x + \bar{i}_y y$$

and

$$\bar{\beta}_{T1} = \bar{i}_x\beta_x + \bar{i}_y\beta_y$$

$$\bar{\beta}_{T2} = \bar{i}_x\beta_x - \bar{i}_y\beta_y$$

$$\bar{\beta}_{T3} = \bar{i}_x\beta_x + \bar{i}_y\beta_y$$

$$\bar{\beta}_{T4} = \bar{i}_x\beta_x - \bar{i}_y\beta_y$$

An alternate and more convenient form of Eq.(II-4.3) is obtained by expanding the exponential using Euler's formula and combining terms so that

$$\bar{E} = \Big[\bar{B}_1 \cos \beta_x x \cos \beta_y y + \bar{B}_2 \cos \beta_x x \sin \beta_y y$$

$$+ \bar{B}_3 \sin \beta_x x \cos \beta_y y + \bar{B}_4 \sin \beta_x x \sin \beta_y y \quad e^{-j\beta_z z} \Big] \qquad \text{(II-4.4)}$$

Note that the $\bar{A}$'s and $\bar{B}$'s are in general complex vectors. For the special case of a rectangular waveguide that is completely filled with an ion-electron beam plasma, Fig.(II-2), the tangential component of electric field must vanish on the walls. Hence from Eq.(II-4.4) we see immediately that in this special case

$$E_x = \Big[B_{2x} \cos \beta_x x \sin \beta_y y + B_{4x} \sin \beta_x x \sin \beta_y y \Big] e^{-j\beta_z z}$$

$$E_y = \Big[B_{3y} \sin \beta_x x \cos \beta_y y + B_{4y} \sin \beta_x x \sin \beta_y y \Big] e^{-j\beta_z z}$$

$$E_z = B_{4z} \sin \beta_x x \sin \beta_y y \quad e^{-j\beta_z z}$$

where $\beta_x = \frac{m\pi}{x_o}$ and $\beta_y = \frac{n\pi}{y_o}$, m and n being non-zero integers, and

$$\beta_T^2 = \left(\frac{m\pi}{x_o}\right)^2 + \left(\frac{n\pi}{y_o}\right)^2 . \qquad \text{(II-4.5)}$$

For the case where the plasma does not completely fill the waveguide, β_x and β_y are not found so easily, since boundary conditions are more complicated.

An obvious solution to the determinantal equation, Eq.(II-4.2), is $\beta_z = \pm\sqrt{k^2 - \beta_T^2}$ which is the same relation that is obtained in the absence of the plasma. This solution applies only to TE modes, since TE modes do not excite any motion of the particles when $B_o = \infty$.

The other solutions to the determinantal equation are roots of the first factor of Eq.(II-4.2). This expression as it stands cannot give β_z explicitly; however, by comparison of the orders of

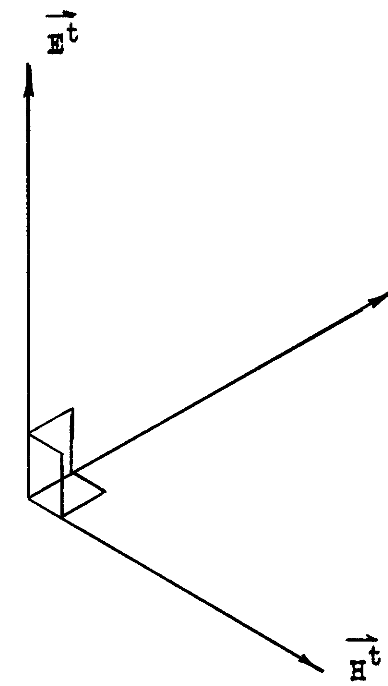


Fig.(II-1). One dimensional case, transverse wave.

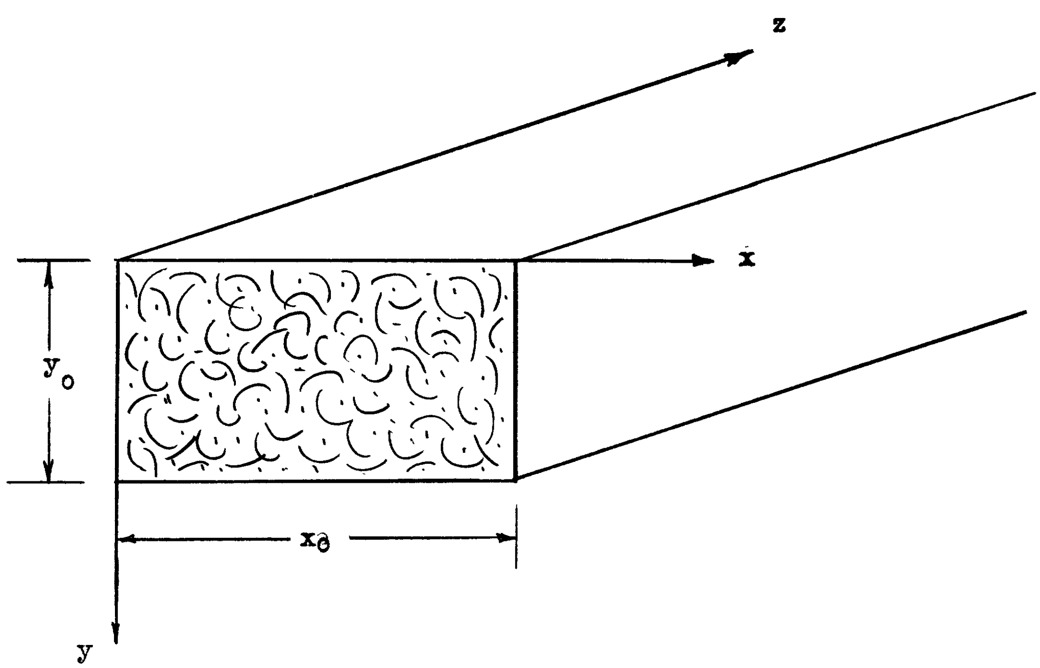


Fig. (II-2). Plasma filled rectangular waveguide.

magnitude of the various terms in the expression and by reasonable assumptions, β_z can be found approximately over certain ranges of frequency.

II-5 Zero Drift and Zero Magnetic Field

For the very special case with $B_o = 0$ and $v_o = 0$, the sixth order determinant, Eq.(II-2.11a), is easily evaluated to give the determinantal equation

$$\left(1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}\right)\left[-\beta_z^2 - \beta_T^2 + k^2\left(1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\bar{\omega}^2}\right)\right] = 0 \qquad \text{(II-5.1)}$$

We see from this equation that for $\omega \neq \sqrt{\omega_{pi}^2 + \omega_{pe}^2}$, waves behave as if they were propagating in a homogeneous, isotropic medium with permitivity

$$\epsilon' = \epsilon_o \left(1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}\right)$$

and for $\omega = \sqrt{\omega_{pi}^2 + \omega_{pe}^2}$, β_z can assume any value at all.

The nature of the propagation is better understood if we examine Maxwell's equations for this special case. Since from Eqs.(I-2.3) to (I-2.6)

$$\rho = \epsilon_o \frac{\omega_{pi}^2 + \omega_{pe}^2}{\omega^2} \nabla \cdot \bar{E}$$

and

$$\bar{J} = -j\omega\epsilon_o \frac{\omega_{pi}^2 + \omega_{pe}^2}{\omega^2} \bar{E},$$

Maxwell's equations, Eqs.(A-1) to (A-4), become

$$\nabla \times \bar{H} = j\omega\epsilon' \bar{E}$$

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}$$

$$\epsilon' \nabla\cdot\bar{E} = 0$$

$$\nabla\cdot\bar{H} = 0$$

For $\omega \neq \sqrt{\omega_{pi}^2+\omega_{pe}^2}$, $\epsilon' \neq 0$ and the equations are the same as obtained for wave propagation in a dielectric of permitivity ϵ'; note here that there is no bunching of charge since $\nabla\cdot\bar{E} = 0$. For $\omega = \sqrt{\omega_{pi}^2+\omega_{pe}^2}$, $\epsilon' = 0$ and $\nabla\cdot\bar{E}$ is in general not zero; at this frequency, since ϵ' vanishes, β_z and β_T^2 are independent with the only restriction that β_T^2 be chosen to satisfy transverse boundary conditions and β_z be chosen to satisfy longitudinal boundary conditions. For a plasma loaded waveguide (the plasma not necessarily filling the guide) of infinite length, β_z can assume any value; if a cavity is made from this wave guide by inserting shorting planes at $z = 0$ and $z = z_0$, then we have the restriction,

$$\beta_z = \pm \frac{\ell\pi}{z_0} \quad , \quad \ell = 1, 2, \ldots..$$

A sketch of β_z as a function of ω for this case of zero magnetic field and zero drift is shown in Fig.(II-3).

II-6 Ions Alone

If we neglect the a.c. effects of the electrons and consider a system consisting of ions alone (the only function of the electron beam being to neutralize the average space charge), then the sixth order determinant, Eq.(II-2.11a), reduces to a third order determinant which is easily expanded to give a determinantal equation that is quadratic in β_z^2. Hence, in this case, β_z can be obtained explicitly as a function of ω.

Because this problem is so much more easily soluble than the total problem, the case of ions alone will be considered in detail in the following chapters with the hope that this approach will give some clue to the solution of the total problem.

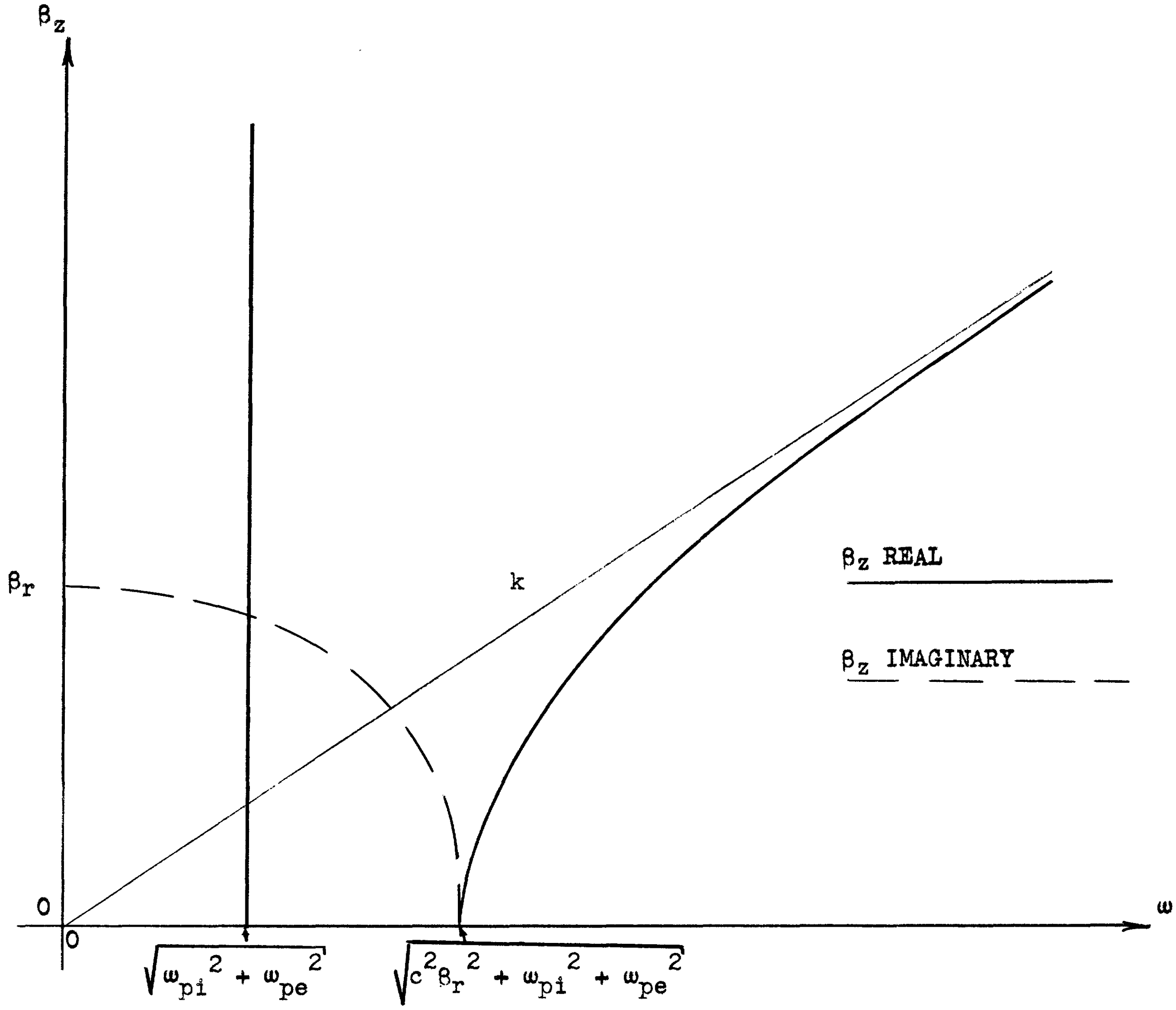


Fig.(II-3). β_z vs. ω, rectangular waveguide filled with plasma, $v_o = 0$, $B_o = 0$.

Chapter III
PROPAGATION IN ION LOADED WAVEGUIDES

III-1 Introduction and Additional Assumptions

As seen in the previous chapter, even the most simplified versions of the total problem lead to expressions so involved that the propagation constants cannot be extracted without resorting to numerical methods. However, if we consider a system containing a plasma of just ions alone, then explicit solutions of the determinantal equation in the plasma can be obtained, because the equation for this case is quadratic in β_z^2; this case is investigated in detail in this chapter because of the simpler mathematics involved. In effect the total problem is reduced to a system with a plasma consisting of trapped ions and an electron beam, the assumed function of the beam being only to neutralize the average dc space charge; the action of the ac electromagnetic fields on the electron beam is ignored - - this amounts to the first step in approaching the total problem on a kinematic basis.

III-2 Field Solution in Ion Plasma for Rectangular Coordinates

For a plasma composed of ions alone, the relations among the small signal velocity components are obtained from Eq. (II-2.11a) when $v_{ex} = v_{ey} = v_{ez} = 0$, and we have

$$\begin{bmatrix} -\beta^2+k^2+\frac{\omega_{pi}^2}{\omega^2}(\beta_x^2-k^2) & \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_y+j\frac{\omega_{ci}}{\omega}(-\beta^2+k^2) & \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_z \\ \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_y-j\frac{\omega_{ci}}{\omega}(-\beta^2+k^2) & -\beta^2+k^2+\frac{\omega_{pi}^2}{\omega^2}(\beta_y^2-k^2) & \frac{\omega_{pi}^2}{\omega^2}\beta_y\beta_z \\ \frac{\omega_{pi}^2}{\omega^2}\beta_y\beta_z & \frac{\omega_{pi}^2}{\omega^2}\beta_x\beta_z & -\beta^2+k^2+\frac{\omega_{pi}^2}{\omega^2}(\beta_z^2-k^2) \end{bmatrix}\begin{bmatrix} v_{ix} \\ v_{iy} \\ v_{iz} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(III-2.1)}$$

The determinant of this matrix equation is easily expanded to give the determinantal equation,

$$(1 - \frac{\omega_{pi}^2}{\omega^2})\left[-\beta_T^2 - \beta_z^2 + k^2 (1 - \frac{\omega_{pi}^2}{\omega^2})\right] - \frac{\omega_{ci}^2}{\omega^2}\left[-\beta_T^2 + (-\beta_z^2 + k^2)(1 - \frac{\omega_{pi}^2}{\omega^2})\right]\left[-\beta_T^2 - \beta_z^2 + k^2\right] = 0 \qquad \text{Eq.(III-2.2)}$$

where $\beta_T^2 = \beta_x^2 + \beta_y^2$ is yet to be determined by boundary conditions. Equation (III-2.2) can be solved explicitly for β_z by means of the quadratic formula; a few curves of positive β_z (forward travelling waves) are plotted numerically in Figs.(III-1) and (III-2) for a rectangular waveguide completely filled with an ion plasma.

From these plots we see there are two waves one of which is cutoff from zero frequency to approximately the cutoff frequency of the empty waveguide (this wave very closely resembles the cutoff waves of an empty* waveguide). The other wave exhibits the same high frequency behaviour, but at low frequencies, well below the cutoff frequency of the empty waveguide, there are two pass bands introduced; propagation is obtained on the approximate intervals $0 < \omega < \omega_{ci}$ and $\omega_{pi} < \omega < \sqrt{\omega_{pi}^2 + \omega_{ci}^2}$ for $\omega_{pi} > \omega_{ci}$, and $0 < \omega < \omega_{pi}$ and $\omega_{ci} < \omega < \sqrt{\omega_{pi}^2 + \omega_{ci}^2}$ for $\omega_{ci} > \omega_{pi}$. These waves are not simple TE or TM modes as found in empty waveguides but are composed of all components of $\bar{\bar{E}}$ and $\bar{\bar{H}}$. Only for the special cases of $B_o = 0$ and ∞ do we get TE and TM waves.

One can proceed from Eq. (III-2.1), the ion velocity coupling equation and from Eq. (I-2.6), the ion force equation, to obtain expressions for the electric field ratios E_x/E_z and E_y/E_z (waves of the form $e^{-j\bar{\bar{\beta}}\cdot\bar{\bar{r}}}$ still assumed). But, it turns out that for the case of ions alone, the field solutions for a circularly symmetric system ($\frac{\partial}{\partial\theta} = 0$) are quite simple - - hence from this point on we consider a circularly cylindrical ion plasma in a circularly cylindrical waveguide and approach the problem directly with all the basic equations

* "empty" meaning in the absence of charge

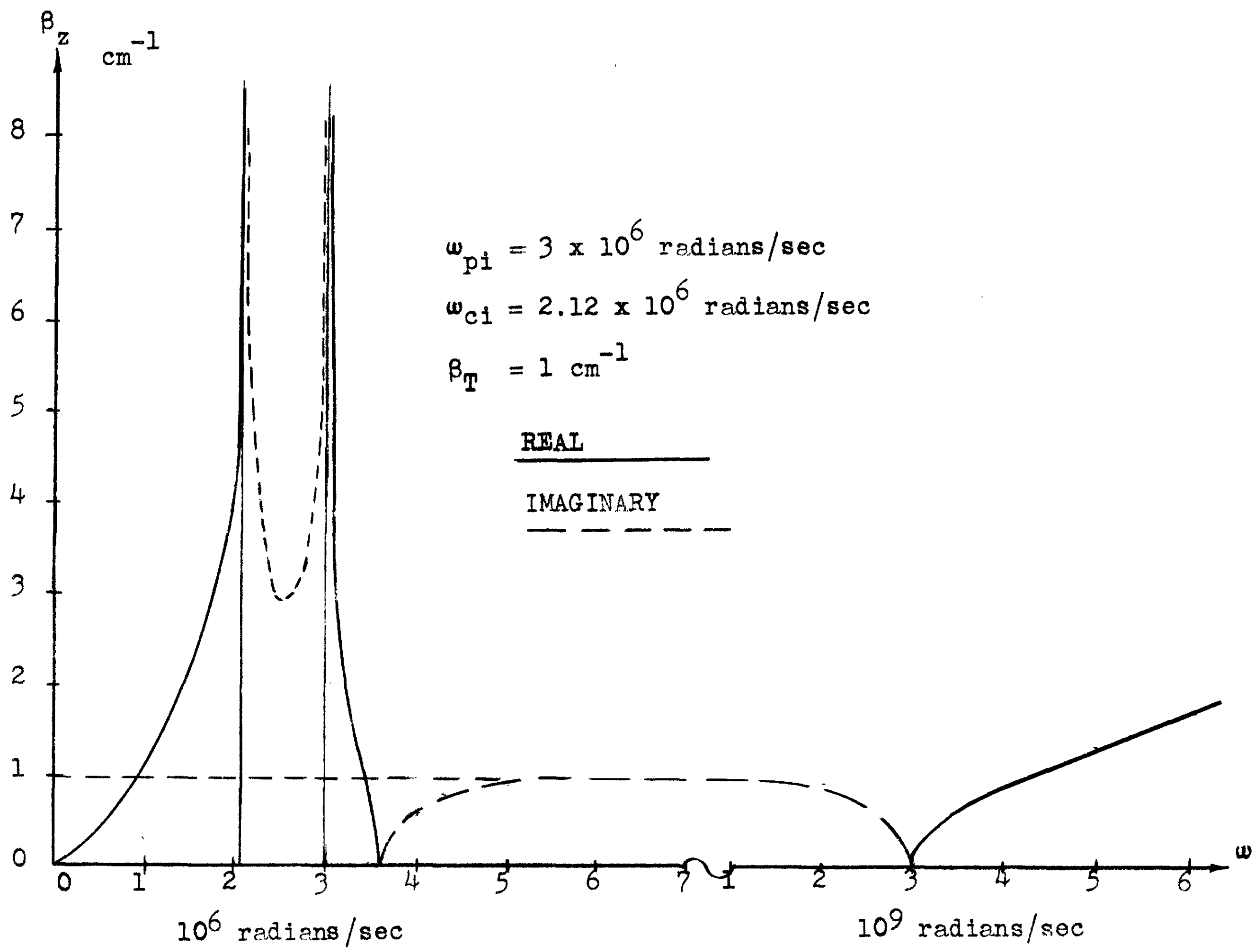


Fig.(III-1). Propagation constant for waves in a rectangular waveguide completely filled with an ion plasma; $\omega_{pi} > \omega_{ci}$. For a given β_T, there are two waves. One is cutoff for low frequencies. The other propagates on two low frequency bands and is cutoff above ω_{pi}. The two waves behave the same above ω_{pi}. Real β_z corresponds to propagation while imaginary β_z corresponds to attenuation.

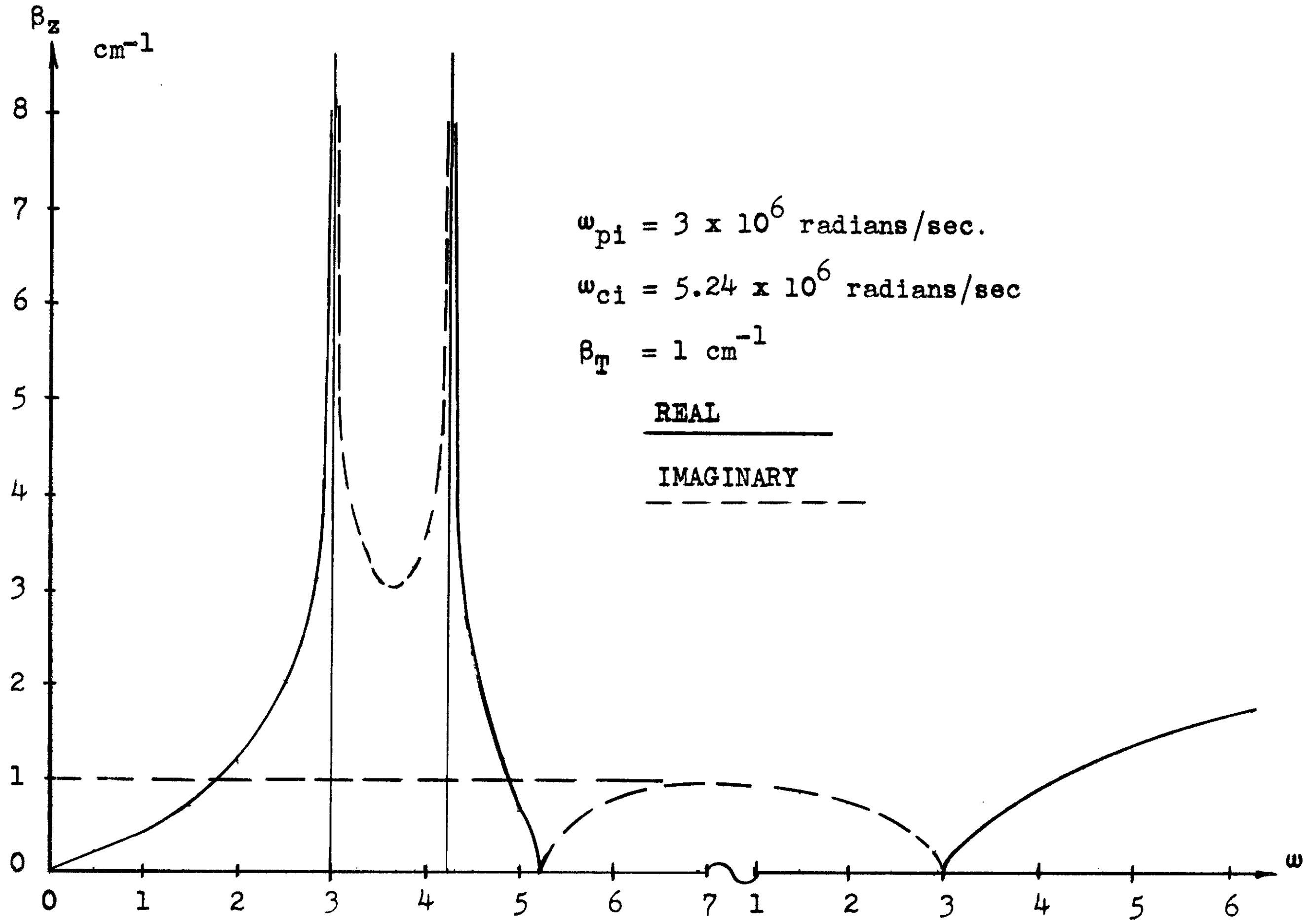


Fig. (III-2). Propagation constant for waves in a rectangular waveguide completely filled with an ion plasma; $\omega_{pi} < \omega_{ci}$. For a given β_T there are two waves. One is cutoff for low frequencies. The other propagates on two low frequency bands and is cutoff above ω_{ci}.

expressed in cylindrical coordinates (r, θ, z) rather than superimpose the rectangular coordinate solutions to synthesize circlarly symmetric fields.

III-3 Field Solutions for Circularly Symmetric Systems.

We now analyze the circularly symmetric system of Fig (III-3); the plasma consists of just ions alone; b is the dc radius of the plasma and a is the radius of the waveguide. The basic equations for a plasma composed only of stationary ions are

$$\nabla^2 \bar{E} + k^2 \bar{E} = j\omega\mu \bar{J} + \nabla(\nabla \cdot \bar{E}) \qquad \text{(III-3.1)}$$

$$\bar{J} = \rho_0 \bar{v}_i \qquad \text{(III-3.2)}$$

$$\frac{j\omega m_i}{q} \bar{v}_i = \bar{E} + \bar{v}_i \times \bar{B}_0 \qquad \text{(III-3.3)}$$

Equation (III-3.1) is the vector wave equation, Eq. (I-2.1), with the quantity ρ/ϵ_0 replaced by its equivalent, $\nabla \cdot \bar{E}$; Eq.(III-3.2) is the definition of ion current density, from Eq. (I-2.4); and Eq.(III-3.3) is the ion force equation, Eq.(I-2.6).

To further simplify the problem, we take advantage of the circular symmetry and look for solutions that are independent of θ $(\frac{\partial}{\partial \theta} = 0)$. From our familiarity with θ-indepent solutions of a circular waveguide we assume that the solutions in a circular cylindrical plasma are of a similar form

$$E_z^{\,I} = AJ_0\,(\beta_r r)\; e^{-j\beta_z z} \qquad \text{(III-3.4)}$$

$$E_r^{\,I} = BJ_1\,(\beta_r r)\; e^{-j\beta_z z} \qquad \text{(III-3.5)}$$

$$E_\theta^{\,I} = CJ_1\,(\beta_r r)\; e^{-j\beta_z z} \qquad \text{(III-3.6)}$$

The superscript I refers to region I, the plasma. In Appendix C, the assumed field, Eq.(III-3.4) to (III-3.6), is shown to be a solution to Eqs.(III-3.1) to (III-3.3) with the results

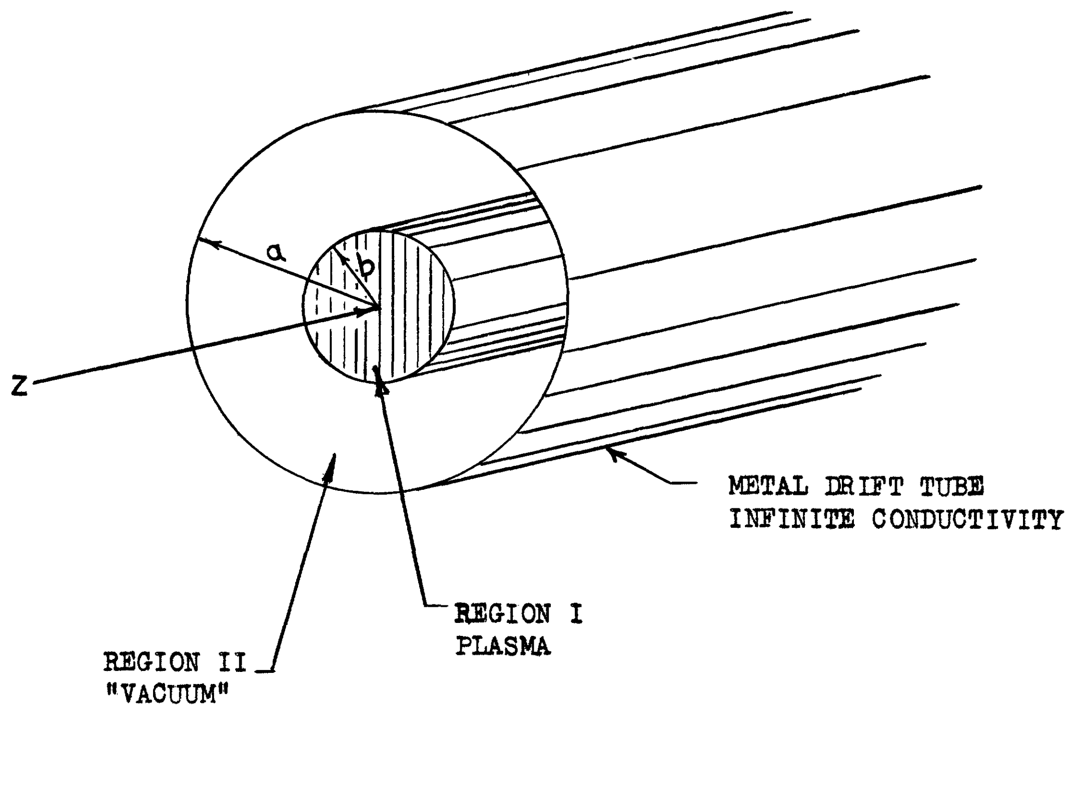


Fig. (III-3). Plasma loaded circular waveguide.

$$E_z^{\,I} = \frac{-j\beta_z\beta_r}{-\beta_r^2+k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)}\; B\, J_0\,(\beta_r r)\; e^{-j\beta_z z} \qquad \text{(III-3.7)}$$

$$E_r^{\,I} = B\, J_1\,(\beta_r r)\; e^{-j\beta_z z}$$

$$E_\theta^{\,I} = \frac{\omega_{ci}}{\omega}\;\frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\;\frac{jk^2}{-\beta_r^2-\beta_z^2+k^2\,\frac{\omega^2-\omega_{pi}^2-\omega_{ci}^2}{\omega^2-\omega_{ci}^2}}\; B\, J_1(\beta_r r)e^{-j\beta_z z} \qquad \text{(III-3.8)}$$

and the determinantal equation

$$\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)\left[-\beta_r^2-\beta_z^2+k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)\right] - \frac{\omega_{ci}^2}{\omega^2}\left[-\beta_r^2 + (-\beta_z^2+k^2)\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)\right]\left[-\beta_r^2-\beta_z^2+k^2\right] = 0 \qquad \text{(III-3.9)}$$

We see that Eq. (III-3.9) is the same determinantal equation as was derived for rectangular coordinates with β_r^2 replacing β_T^2.

III-4 The Quasi-Static Approximation

We assume that the ion plasma and cyclotron frequencies are several orders of magnitude smaller than the cutoff frequency of the empty waveguide; plasma and cyclotron frequencies are of the order of a few magacycles, whereas laboratory size waveguide cutoff frequencies are of the order of several thousands of magacycles. These great differences in frequency allow us to make approximations which greatly simplify the results of the previous section and to introduce a useful concept, the "quasi-static approximation", which applies to low frequency solutions.

The typical ion plasma and cyclotron frequencies mentioned above are for gases such as hydrogen, nitrogen, etc. The approximations about to be developed also apply to an ion plasma composed of electrons rather than ionized gas molecules if the electron plasma and cyclotron frequencies are small compared with the empty waveguide cutoff. All equations until now are equally applicable to both stationary ion plasmas consisting of gas ions and those consisting of electrons with proper regard to the sign of the electron charge.

For $\omega >> \omega_{pi}$ and ω_{ci}, the determinantal equation, Eq. (III-3.9) reduces to

$$-\beta_r^2 - \beta_z^2 + k^2 = 0 \qquad \text{(III-4.1)}$$

which is the same determinantal equation that is obtained in the absence of the plasma. From. Eq. (III-4.1), the field solutions in the plasma, Eqs. (III-3.5),(III-3.7),(C-11), become for $\omega >> \omega_{pi}$ and ω_{ci}

$$E_z^I = -j \frac{\beta_r}{\beta_z} B\ J_o\ (\beta_r r)\ e^{-j\beta_z z} \qquad \text{(III-4.2)}$$

$$E_r^I = B\ J_1\ (\beta_r r)\ e^{-j\beta_z z} \qquad \text{(III-4.3)}$$

$$H_\theta^I = \frac{\omega\epsilon_o}{\beta_z} B\ J_1\ (\beta_r r)\ e^{-j\beta_z z} \qquad \text{(III-4.4)}$$

E_θ^I, H_r^I, and H_z^I vanish. Equations (III-4.1) to (III-4.4) are exactly the solution of the radially symmetric modes found in circular waveguides; hence for $\omega >> \omega_{pi}$ and ω_{ci}, the plasma has no effect on the propagation and a waveguide containing an ion plasma supports the same waves at high frequencies as in the absence of the plasma.

The low frequencies are of greater interest since the plasma oscillations are observed at low frequencies. For low frequencies (well below the empty waveguide cutoff) we make the assumption that k^2 can be neglected in comparison with β_z^2 and β_r^2. With this assumption, the roots of the determinantal equation, Eq. (III-3.9), reduce to the compact expressions

$$\beta_z^2 = \frac{\omega^2(\omega_{pi}^2 + \omega_{ci}^2 - \omega^2)}{(\omega_{pi}^2 - \omega^2)\ (\omega_{ci}^2 - \omega^2)}\ \beta_r^2 \qquad \text{(III-4.5)}$$

and

$$\beta_z^{\,2} = -\beta_r^{\,2} \qquad \text{(III-4.6)}$$

Equation (III-4.6) resembles the empty waveguide modes and is not of interest since these modes are cutoff for low frequencies. Eq.(III-4.5) is the interesting root of the determinantal equation since it exhibits the two low frequency pass bands discovered in numerically solving for the roots of the exact determinantal equation, Eq. (III-2.2). We note that if we choose $\beta_r^{\,2} >> k^2$, Eqs.(III-4.5) and (III-4.6) are consistent with the assumption that $|\beta_z^{\,2}| >> k^2$. Equation(III-4.5) is sketched in Figs.(III-4) and (III-5).

With the same assumption that $k^2 << |\beta_z^{\,2}|$ and $|\beta_r^{\,2}|$, the electric field in the plasma becomes

$$E_z^{\,I} = j\,\frac{\beta_z}{\beta_r}\,BJ_o\,(\beta_r r)\,e^{-j\beta_z z} \qquad \text{(III-4.7)}$$

$$E_r^{\,I} = B\,J_1\,(\beta_r r)\,e^{-j\beta_z z} \qquad \text{(III-4.8)}$$

We assume that

$$E_\theta^{\,I} = 0 \qquad \text{(III-4.9)}$$

since from Eq. (III-3.8), $E_\theta^{\,I}$ is proportional to k^2.

We see from Eqs.(III-4.7) to (III-4.9) that

$$\bar{\bar{E}}^I = -\left[\bar{i}_r\,\frac{\partial}{\partial r} + \bar{i}_z\,\frac{\partial}{\partial z}\right]\frac{B}{\beta_r}\,J_o\,(\beta_r r)\,e^{-j\beta_z z}$$

or if we let

$$V^I = \frac{B}{\beta_r}\,J_o\,(\beta_r r)\,e^{-j\beta_z z}$$

we have

$$\bar{\bar{E}}^I = -\nabla V^I; \qquad \text{(III-4.10)}$$

this expression is the quasi-static approximation.

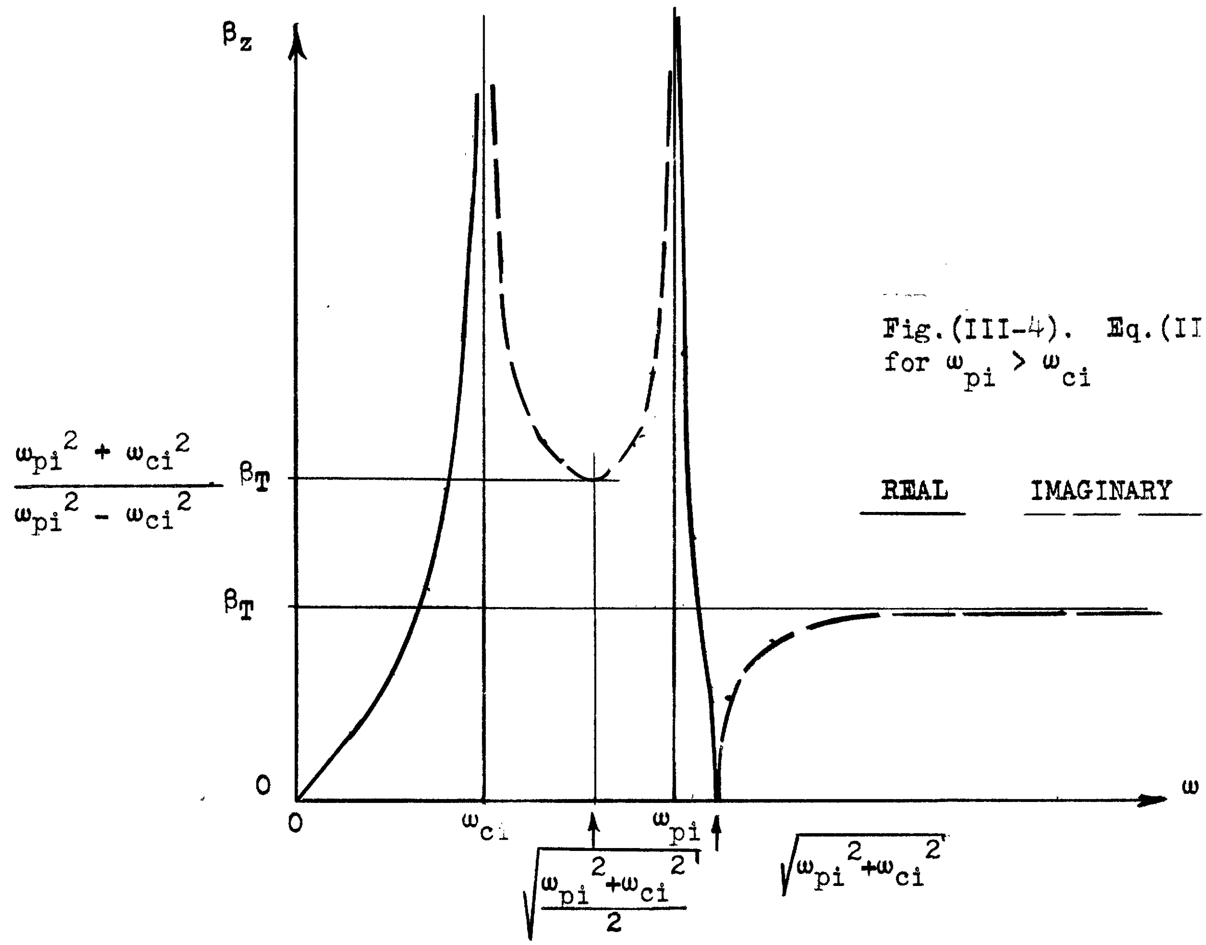


Fig.(III-4). Eq.(III-4.5) for $\omega_{pi} > \omega_{ci}$

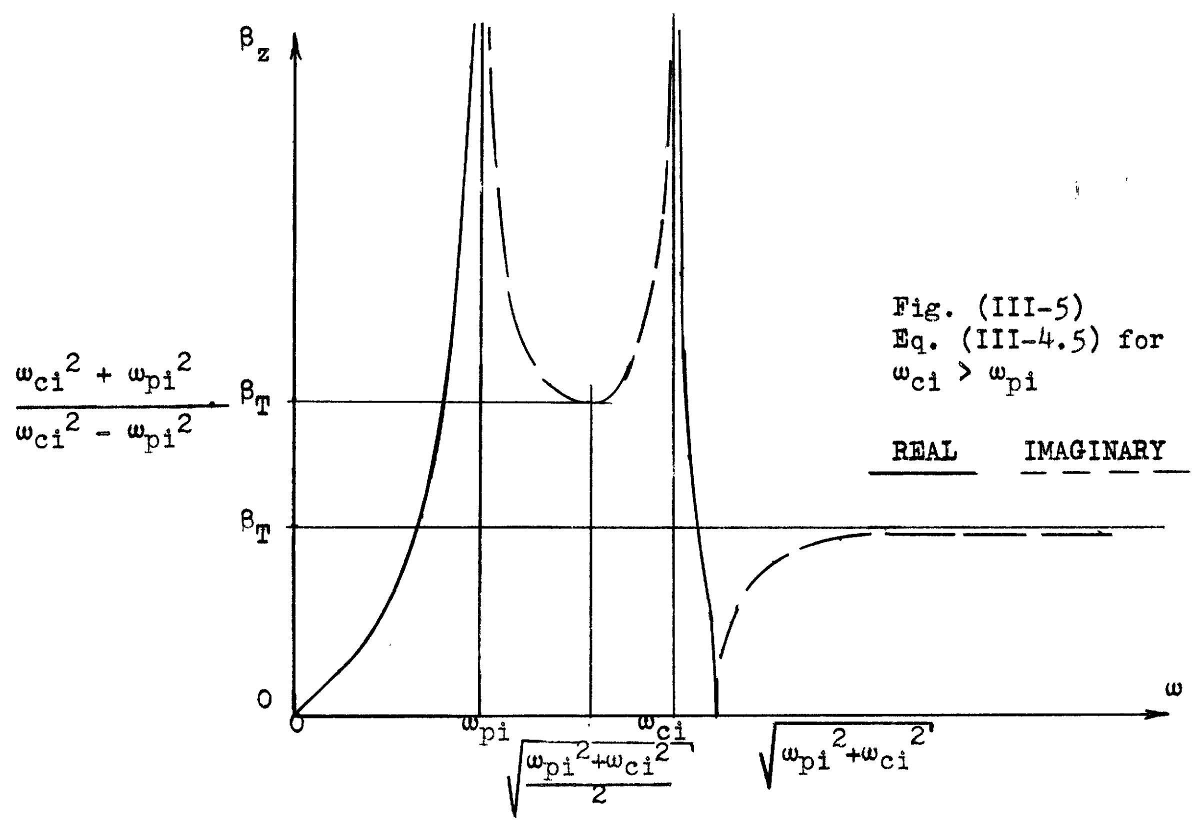


Fig. (III-5) Eq. (III-4.5) for $\omega_{ci} > \omega_{pi}$

Using the quasi-static approximation we reformulate the basic equations for low frequencies. Instead of the vector wave equation, Eq.(III-3.1), we use Poisson's Equation

$$\nabla^2 V = -\rho/\epsilon_o \qquad \text{(III-4.11)}$$

Equations (III-4.10) and (III-4.11), together with the ion force equation, Eq. (III-3.3), the current-velocity equation, Eq.(III-3.2), and the ion continuity equation,

$$\nabla \cdot J + j\omega\rho = 0,$$

give us the same results as the exact solutions, Eqs.(III-4.5), (III-4.7) and (III-4.8) with the assumption that $k^2 << \beta_r^2$ and $|\beta_z^2|$; this point is illustrated in Appendix D.

The above steps are by no means a proof of the quasi-static approximation formulation. Having made the reasonable assumption that k^2 is negligible for the frequencies of interest, we noticed that the simplified expressions for the components of $\bar{E}$ suggested that $\bar{E} = -\nabla V$. Now from the hypothesis that $\bar{E} = -\nabla V$ and the assumption that electrostatic expressions were applicable, it was shown (Appendix D) that the results of using the quasi-static approximation independently were the same as the approximations obtained from simplifying the exact solutions.

Also developed in Appendix D is the quasi-static θ - component of the magnetic field; we are concerned with H_θ, since only H_θ contributes to the z - directed power flow in the plasma. The quasi-static E_r and H_θ may be expressed as

$$\frac{\partial E_r^I}{\partial z} = -j \frac{\beta_r^2}{\omega\epsilon_o} \frac{\omega^2}{\omega_{pi}^2 - \omega^2} H_\theta^I \qquad \text{(III-4.12)}$$

$$\frac{\partial H_\theta^I}{\partial z} = -j\omega\epsilon_o \frac{\omega_{pi}^2 + \omega_{ci}^2 - \omega^2}{\omega_{ci}^2 - \omega^2} E_r^I \qquad \text{(III-4.13)}$$

which are analagous to the transmission line equations

$$\frac{\partial V}{\partial z} = -Z\ I$$

$$\frac{\partial I}{\partial z} = -Y\ V$$

Z and Y being the series impedance and shunt admittance, respectively, per unit length of transmission line.

The circuit analogue for Eqs.(III-4.12) and (III-4.13) is given in Fig.(III-6). In the absence of ions ($\omega_{pi}^2 = 0$) the circuit reduces to the series capacitance $\frac{\epsilon_o}{\beta_r^2 dz}$ and the shunt capacitance $\epsilon_o dz$ which is the circuit analogue of a cutoff radially symmetric empty waveguide mode. The ions, in effect, add inductances so that a waveguide can become a propagating structure at low frequencies, a series positive reactance and a shunt negative reactance (or vice versa) being necessary for a transmission line to support propagation.

From transmission line theory we know that if the z - dependence $e^{-\Gamma z}$ is assumed, then

$$\Gamma^2 = Z\ Y$$

If we substitute the equivalent Z and Y of Eqs. (III-4.12) and (III-4.13) into this expression we arrive at the result that

$$\Gamma^2 = -\beta_z^2$$

This result is important for it shows that the first order magnetic field is consistent with the quasi-static approximation. Recall that since it was assumed $\bar{\bar{E}}^I = -\nabla V$, then $\nabla \times \bar{\bar{E}} = 0$ from which $\bar{\bar{H}} = 0$; but having found our assumed $\bar{\bar{E}}$ consistent with the set of low frequency equations, Eq.(D-1) to (D-5), we proceeded to calculate H_θ as a first order perturbation of our assumed solution through Ampere's Circuital Law (Appendix D).

The characteristic impedance of a transmission line is given by

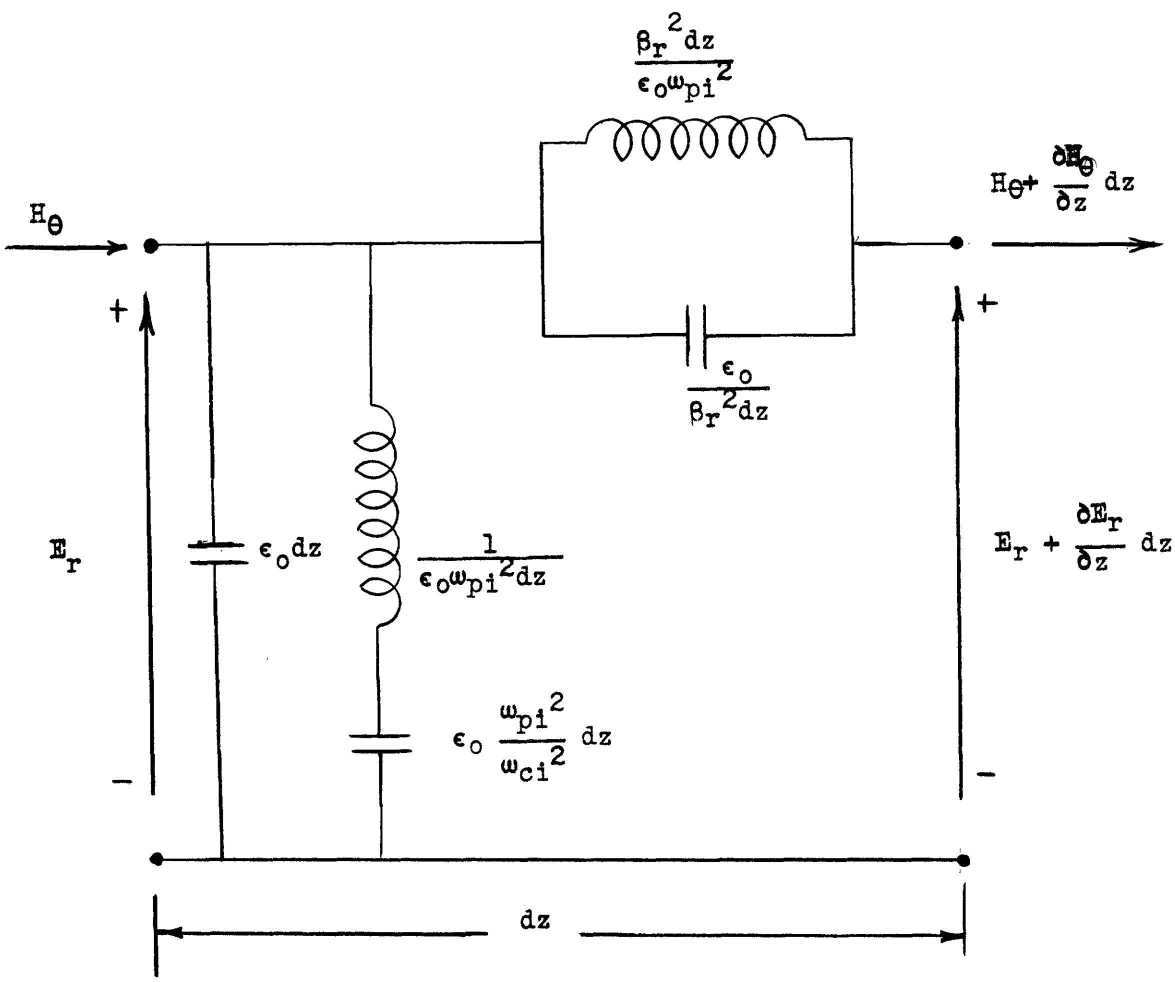


Fig. (III-6). Circuit analogue for quasi-static plasma waves.

$Z_o = \sqrt{Z/Y}$. Hence for the radial ion modes

$$Z_o = \frac{1}{\omega\epsilon_o} \frac{\omega^2}{\omega_{pi}{}^2-\omega^2} \frac{\beta_r{}^2}{\beta_z} \qquad \text{(III-4.14)}$$

which is real for β_z real (in pass bands) and imaginary for β_z imaginary (in attenuation bands). If we have a system with only a forward travelling wave in the plasma, the z - component of Poynting's vector is given by

$$S_z = \frac{1}{2} E_r H_\theta^* = \frac{1}{2} Z_o |H_\theta|^2 \qquad \text{(III-4.15)}$$

Here we see that real power flows in the z-direction in the plasma when β_z is real. For β_z real and $\omega < \omega_{pi}$, Z_o is positive - hence the power flow is in the positive z-direction. For β_z real and $\omega > \omega_{pi}$ we see from Eq. (III-4.14) that Z_o is negative - hence the power flow is in the negative z-direction; this occurs because the group velocity

$$\frac{\partial\omega}{\partial\beta_z} < 0$$

for $\omega_{pi}{}^2 + \omega_{ci}{}^2 > \omega^2 > \omega_{pi}{}^2$ when $\omega_{pi} > \omega_{ci}$ and for $\omega_{pi}{}^2 + \omega_{ci}{}^2 > \omega^2 > \omega_{ci}{}^2$ when $\omega_{ci} > \omega_{pi}$ (See Figs.(III-4) and (III-5)).

The quasi-static approximation seems to give reliable results. Numerical computations of the propagation constant from the quasi-static expression agree for several decimal places with computations from the exact determinantal equation, Eq.(III-2.2). In the following sections we consider specific boundary value problems and make use of the quasi-static approximation.

III-5 Quasi-Static Ion Filled Waveguide

For a waveguide that is completely filled with an ion plasma $\left[b = a \text{ in Fig.(III-3)}\right]$, the only boundary condition to be met is that the quasi-static potential vanish on the waveguide walls. As shown in the previous section for the radially symmetric plasma waves

$$V(r,z) = V_o J_o(\beta_r r)\, e^{-j\beta_z z}$$

The boundary condition $V(b,z) = 0$ requires that $J_0\ (\beta_r b) = 0$ -- hence $\beta_r b = 2.405,\ 5.520,\ 8.654$, etc., the roots of the zero order Bessel function. For a given drift tube radius then, $\beta_r b$ can assume an infinite number of discrete values, thus giving us an infinite number of radially symmetric modes; in Fig.(III-7) the transverse variation of $V\ (r,z)$ is sketched for a few values of $\beta_r b$. The dispersion relation for the ion filled waveguide is given by the ion plasma determinantal equation, Eq.(III-4.5) where β_r takes on the discrete values described above.

III-6 Quasi-Static Waveguide Partly Filled with Ions

The analysis of the waveguide that is partly filled with a non-drifting plasma $\left[\ b < a \text{ as in Fig.(III-3)}\right]$ is not so simple as the completely filled waveguide since the more involved boundary conditions result in transcendental equations that must be solved simultaneously with Eq. (III-4.5) to obtain the dispersion relation. The quantity β_r for this case is a function of frequency, a fact which further complicates the situation. No numerical solutions for this case have been computed because of the tedious labor that would be involved. Rather here, the equations are analyzed and the dispersion relations are sketched qualitatively. Such a qualitative analysis - if it is possible, as is the case here - is more valuable than a lot of computed curves since it brings more into evidence the general behaviour.

In this case we must consider the fields in the vacuum, region II. The fields in the vacuum are described by the quasi-static equations

$$\bar{E}^{II} = -\nabla V^{II} \qquad \text{(III-6.1)}$$

$$\nabla^2 V^{II} = 0 \qquad \text{(III-6.2)}$$

In cylindrical coordinates the solution to Laplace's equation, Eq. (III-6.2), for circularly symmetric fields is

$$V^{II} = \left[A\ I_0\ (\beta_z r) + B\ K_0(\beta_z r)\right]\ e^{-j\beta_z z} \qquad \text{(III-6.3)}$$

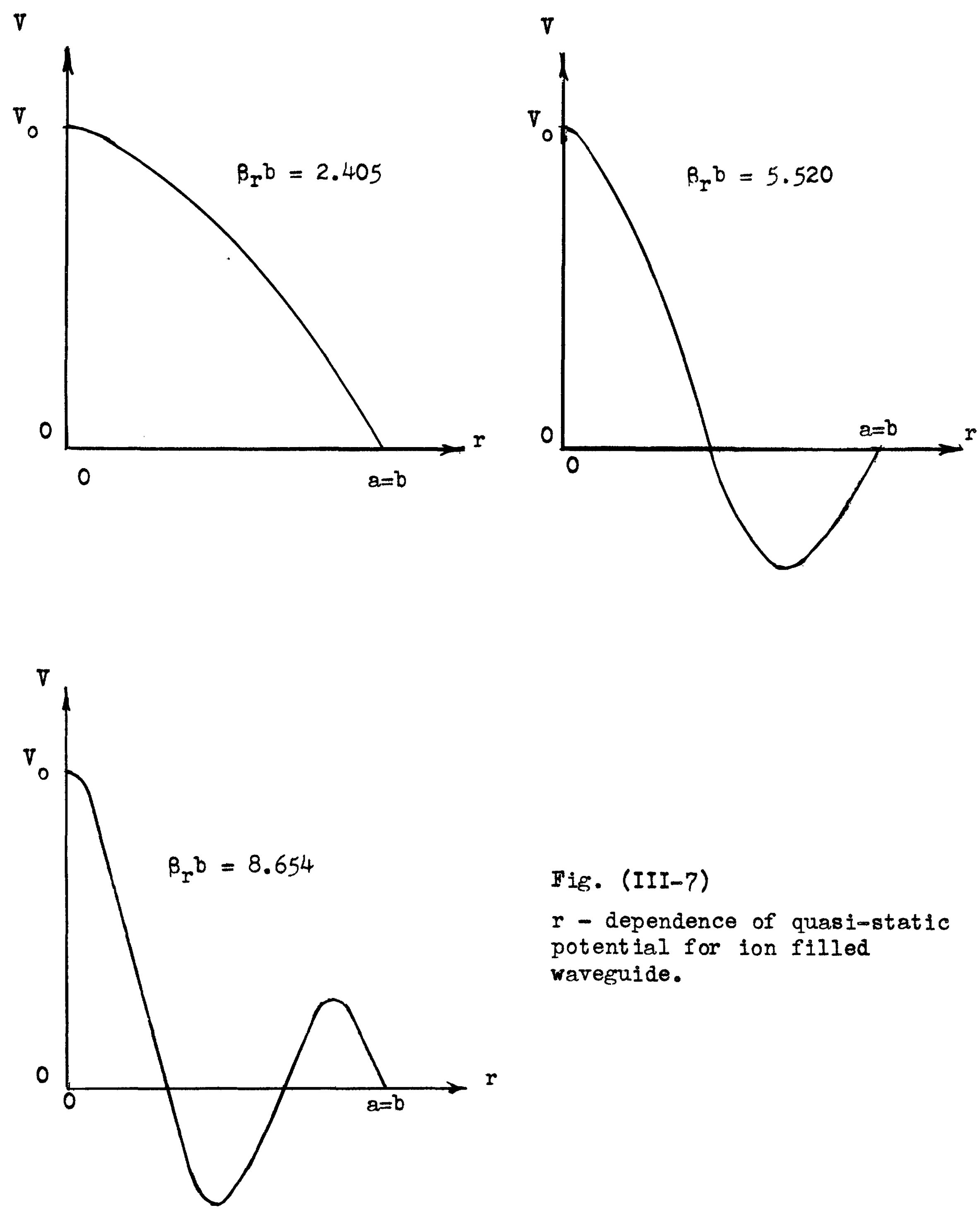


Fig. (III-7)

r - dependence of quasi-static potential for ion filled waveguide.

where A and B are constants yet to be determined by boundary conditions and I_o and K_o are the zero order modified Bessel functions of the first and second kind, respectively. From Eqs. (III-6.1) and (III-6.3) the electric field components in the vacuum are

$$E_r^{II} = -\beta_z \left[A\, I_1 (\beta_z r) - B\, K_1 (\beta_z r)\right] e^{-j\beta_z z} \quad \text{(III-6.4)}$$

$$E_z^{II} = j\beta_z \left[A\, I_o (\beta_z r) + B\, K_o (\beta_z r)\right] e^{-j\beta_z z} \quad \text{(III-6.5)}$$

At r = a we have the boundary condition that $E_z^{II} = 0$; this is equivalent to

$$V^{II} (a) = 0 \quad \text{(III-6.6)}$$

At r = b the boundary condition that $E_z^{II} (b) = E_z^{I} (b)$ is equivalent to

$$V^{II} (b) = V^{I} (b) \quad \text{(III-6.7)}$$

In the plasma we have seen that

$$V^{I} = V_o J_o (\beta_r r)\, e^{-j\beta_z z} \quad ;$$

hence from this expression and the boundary conditions Eqs.(III-6.6) and (III-6.7) the constants A and B are evaluated with the result

$$V^{II} = V_o J_o (\beta_r b)\, \frac{I_o(\beta_z r) - \dfrac{I_o(\beta_z a)}{K_o(\beta_z a)}\, K_o(\beta_z r)}{I_o(\beta_z b) - \dfrac{I_o(\beta_z a)}{K_o(\beta_z a)}\, K_o(\beta_z b)}\, e^{-j\beta_z z} \quad \text{(III-6.8)}$$

The negative of the gradient of V^{II} yields

$$E_z^{II} = j\beta_z V^{II} \quad \text{(III-6.9)}$$

$$E_r^{II} = -\beta_z V_o J_o(\beta_r b) \frac{I_1(\beta_z r) + \frac{I_o(\beta_z a)}{K_o(\beta_z a)} K_1(\beta_z r)}{I_o(\beta_z b) - \frac{I_o(\beta_z a)}{K_o(\beta_z a)} K_o(\beta_z b)} e^{-j\beta_z z} \qquad \text{(III-6.10)}$$

At $r = b$ we have the additional boundary condition that

$$E_r^{II}(b) = E_r^{I}(b) + \sigma/\epsilon_o \qquad \text{(III-6.11)}$$

from Eq. (I-3.4). The equivalent surface charge density is given by

$$\sigma = \left.\frac{\rho_o v_{ir}}{j\omega}\right|_{r=b} = \left.\frac{J_{ir}}{j\omega}\right|_{r=b}$$

from Eq. (I-3.1); but, since from Eq. (C-3)

$$J_{ir} = j\omega\epsilon_o \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} E_r^{I}$$

we have from Eq. (III-6.11)

$$E_r^{II}(b) = \left[1 + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2}\right] E_r^{I}(b) \qquad \text{(III-6.12)}$$

The substitution of Eqs. (III-6.10) and (III-4.8) into (III-6.12) yields the boundary matching equation

$$\frac{\omega_{pi}^2 + \omega_{ci}^2 - \omega^2}{\omega_{ci}^2 - \omega^2} \beta_r b \frac{J_1(\beta_r b)}{J_o(\beta_r b)} =$$

$$-\beta_z b \frac{I_1(\beta_z b) + \frac{I_o(\beta_z a)}{K_o(\beta_z a)} K_1(\beta_z b)}{I_o(\beta_z b) - \frac{I_o(\beta_z a)}{K_o(\beta_z a)} K_o(\beta_z b)} \qquad \text{(III-6.13)}$$

This equation and the now familiar plasma determinantal equation, Eq. (III-4.5), when solved simultaneously, give the dispersion, β_z as a function of ω, for the waveguide partly filled with ions.

Before further discussion, we must recognize that the plasma potential, $V^I = V_o J_o(\beta_r r) e^{-j\beta_z z}$, is not the only circularly symmetric solution to the set of quasi-static plasma equations, Eqs.(D-1) to (D-5). The potential function

$$V^I = V_o I_o (\alpha_r r) e^{-j\beta_z z} \tag{III-6.14}$$

is a bonafide solution of our system; this solution gives the plasma determinantal equation

$$\beta_z^2 = -\frac{\omega^2(\omega_{pi}^2+\omega_{ci}^2-\omega^2)}{(\omega_{pi}^2-\omega^2)(\omega_{ci}^2-\omega^2)} \alpha_r^2 \tag{III-6.15}$$

which differs only in the minus sign from Eq. (III-4.5). This solution, the modified Bessel function solution, pertains to what is called a "surface wave" and the ordinary Bessel function solution pertains to a "body wave".[8] In the surface wave the ac charge is most dense at the surface of the plasma, whereas, in the body wave, the maximum ac charge density occurs at the center of the plasma.

In effect, all relations concerned with the surface wave are found from the body wave relatbns through the substitution $\beta_r = j\alpha_r$. Recalling that $J_o(jx) = I_o(x)$ and $J_1(jx) = jI_1(x)$, we obtain the similar relations for the surface wave from Eqs. (III-6.8) and (III-6.13),

$$V^{II} = V_o I_o(\alpha_r b) \frac{I_o(\beta_z r) - \frac{I_o(\beta_z a)}{K_o(\beta_z a)} K_o(\beta_z r)}{I_o(\beta_z b) - \frac{I_o(\beta_z a)}{K_o(\beta_z a)} K_o(\beta_z b)} e^{-j\beta_z z} \tag{III-6.16}$$

$$-\frac{\omega_{pi}^2 + \omega_{ci}^2 - \omega^2}{\omega_{ci}^2 - \omega^2} \, \alpha_r b \frac{I_1(\alpha_r b)}{I_0(\alpha_r b)} = -\beta_z b \frac{I_1(\beta_z b) + \frac{I_0(\beta_z a)}{K_0(\beta_z a)} K_1(\beta_z b)}{I_0(\beta_z b) - \frac{I_0(\beta_z a)}{K_0(\beta_z a)} K_0(\beta_z b)} \tag{III-6.17}$$

The problem now is to obtain β_z as a function of ω from Eqs. (III-4.5) and (III-6.13) for the case of the body wave, and from Eqs. (III-6.15) and (III-6.17) for the case of the surface wave; curves which aid in the analysis are sketched in Figs. (III-8), to (III-10). Both the right hand side of Eq. (III-6.15) and the right hand side of Eq. (III-6.17) must be positive in order for the surface wave to propagate (β_z real); also, the right hand side of Eq. (III-6.17) is always greater than $\beta_z b$ for <u>a</u> finite [Fig. (III-8)] and $\alpha_r b \frac{I_1(\alpha_r b)}{I_0(\alpha_r b)}$ is always less than $\alpha_r b$ [Fig. (III-10)]: from this information it is easily shown that <u>the surface wave propagates for $\omega_{pi} > \omega_{ci}$ only and in the frequency range $\omega_{ci} \leq \omega \leq \sqrt{\frac{\omega_{pi}^2 + \omega_{ci}^2}{2}}$.</u> Similarly, both the right hand side of Eq. (III-4.5) and the right hand side of Eq. (III-6.13) must be positive in order for the body wave to propagate: from the nature of these equations, we conclude that <u>the body wave propagates in the frequency ranges $0 \leq \omega \leq \omega_{ci}$ and $\omega_{pi} \leq \omega \leq \sqrt{\omega_{pi}^2 + \omega_{ci}^2}$ for $\omega_{pi} > \omega_{ci}$; and in $0 \leq \omega \leq \omega_{pi}$ and $\omega_{ci} \leq \omega \leq \sqrt{\omega_{pi}^2 + \omega_{ci}^2}$ for $\omega_{ci} > \omega_{pi}$.</u>

A careful study of Eqs. (III-4.5), (III-6.13), (III-6.15), and (III-6.17) in conjunction with Figs. (III-4), (III-5), (III-8), (III-9), and (III-10) finally yields the propagation constant β_z as a function of ω as sketched in Figs. (III-11) and (III-12). Here we see that β_z vs. ω has approximately the same character as the filled waveguide case except for the introduction of the surface wave. Note that the surface wave appears only for the lowest circularly symmetric mode since the left hand side of Eq. (III-6.17) is not repetitive as is the left hand side of Eq.(III-6.13); the repetitive nature of Eq.(III-6.13) accounts for the infinite set of modes, all of which are body waves. At $\omega = \omega_{ci}$, there is a continuous transition in the lowest radial mode from the body wave to the surface wave. Several potential variations for the partly filled waveguide are sketched in Fig.(III-13).

III-7 Summary of Chapter

In this chapter the properties of waveguides loaded with a stationary ion plasma were studied. It was found that at high frequencies, frequencies comparable to the empty waveguide cutoff, the waves behaved as if the plasma were absent. For low frequencies it was demonstrated that a very simplified approach to the problem, the quasi-static approach, gave solutions which reasonably approximated the exact solutions. It was found that the presence of the plasma enabled a waveguide which would be ordinarily cutoff to become a propagating structure in two low frequency bands.

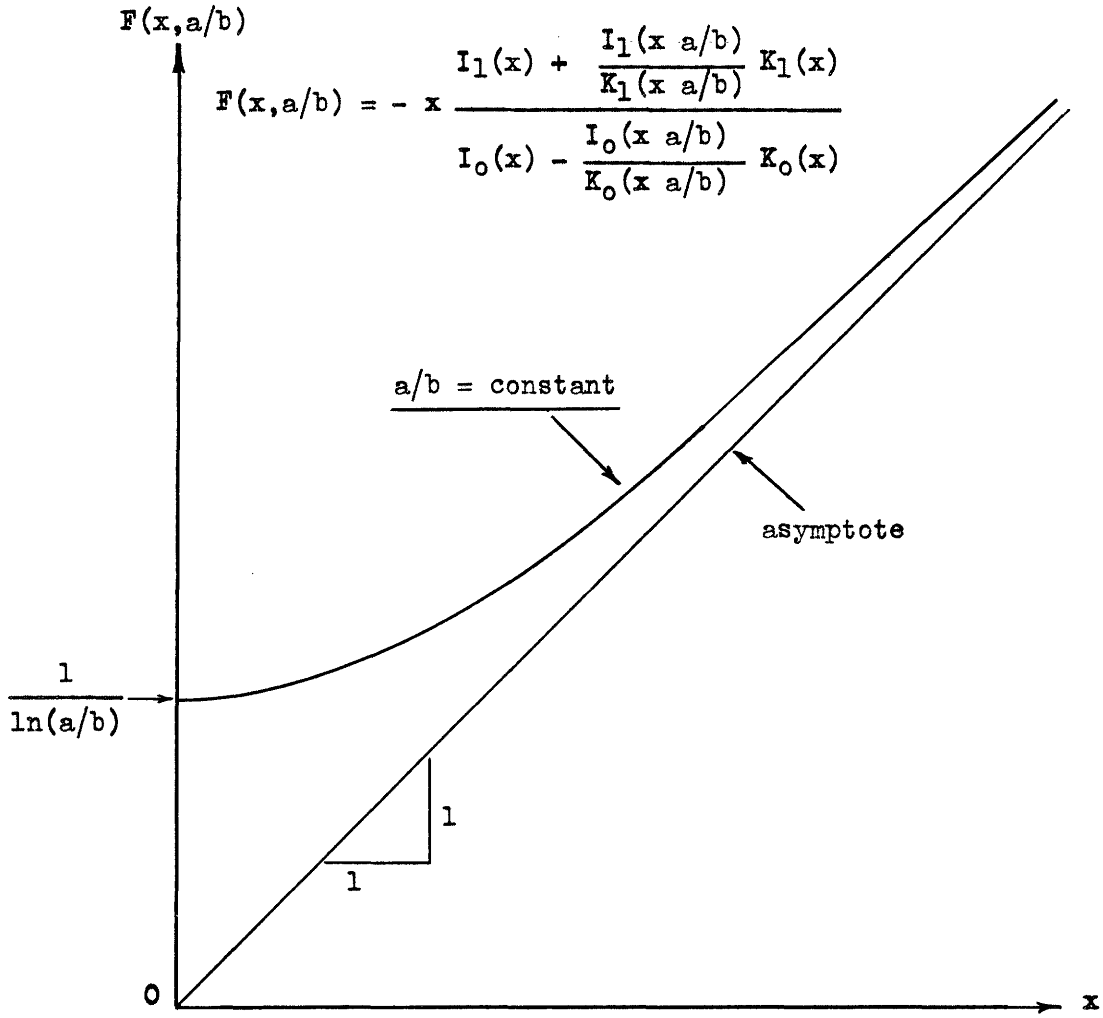


Fig.(III-8). Sketch of right hand side of Eqs.(III-6.13) and (III-6.17); $x = \beta_z b$. As $x \to \infty$, $F(x,a/b) \to x$. For $a/b = \infty$, $F(x,a/b) = x$.

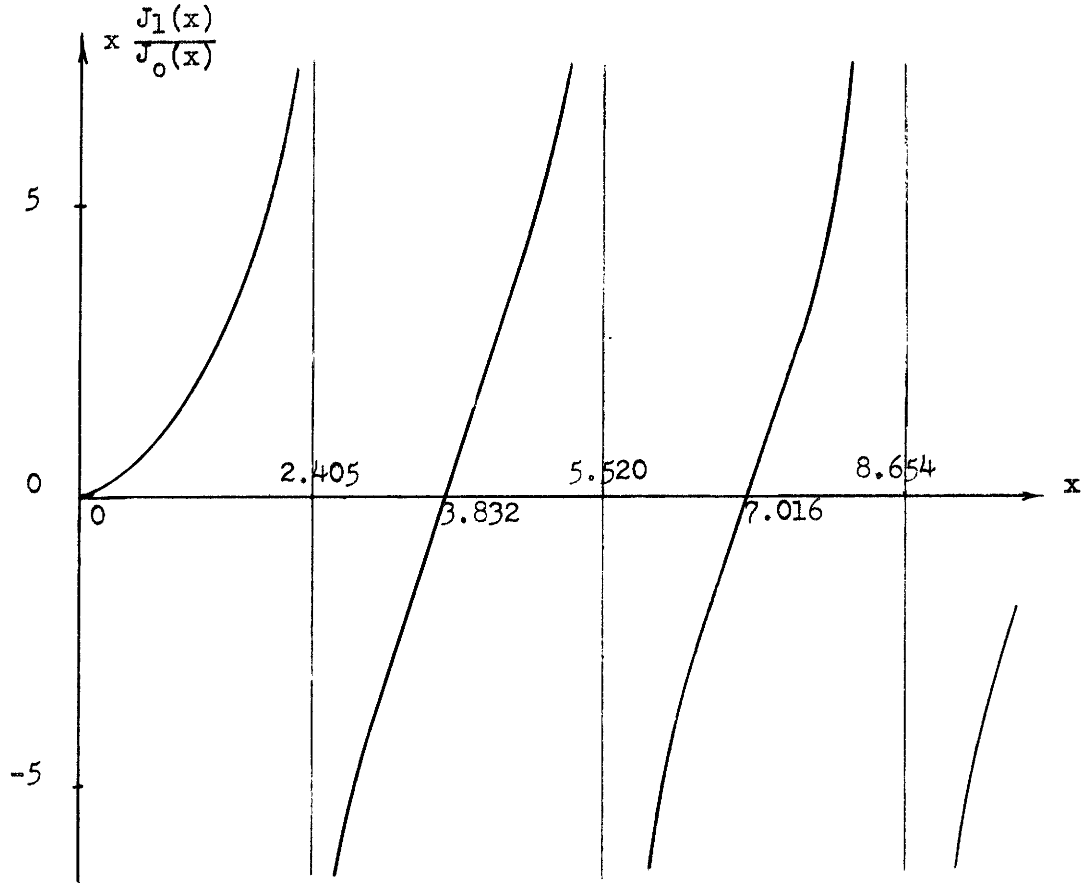


Fig.(III-9). Sketch of factor in left hand side of Eq.(III-6.13); $x = \beta_r b$.

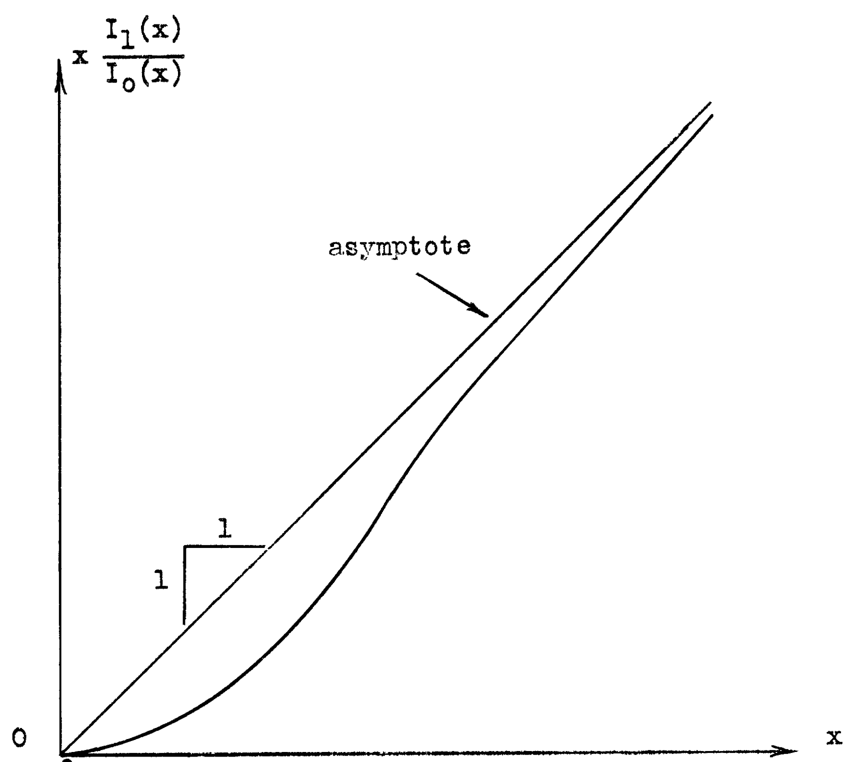


Fig.(III-10). Sketch of factor in left hand side of Eq.(III-6.17); $x = \alpha_r b$.

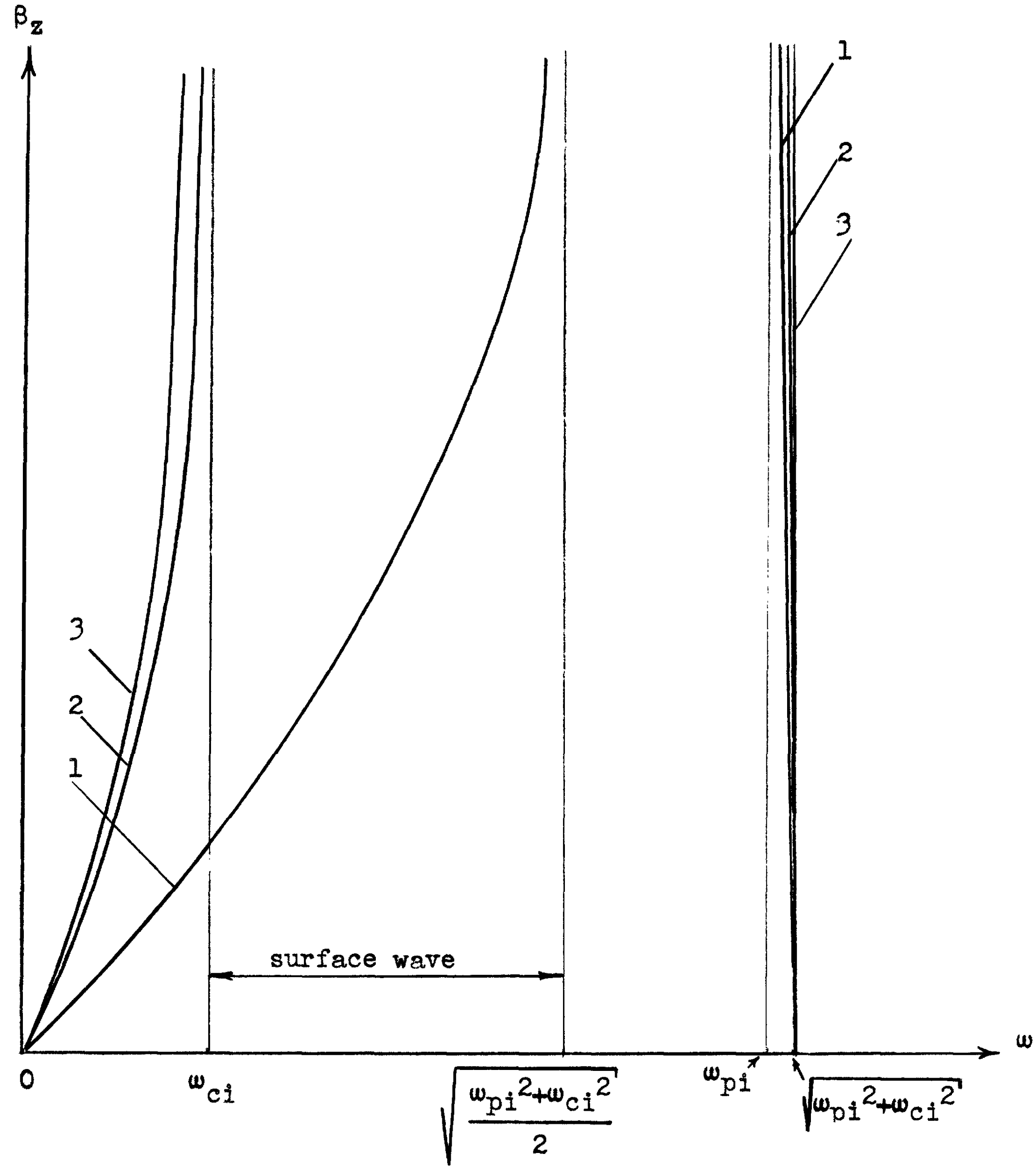


Fig.(III-11). Sketch of ω - β diagram for $b/a < 1$, $\omega_{pi} > \omega_{ci}$. The numbers indicate the order of the radial mode. All modes > 1 have the same pass bands. The surface wave exists where indicated; the rest of the diagram corresponds to body waves.

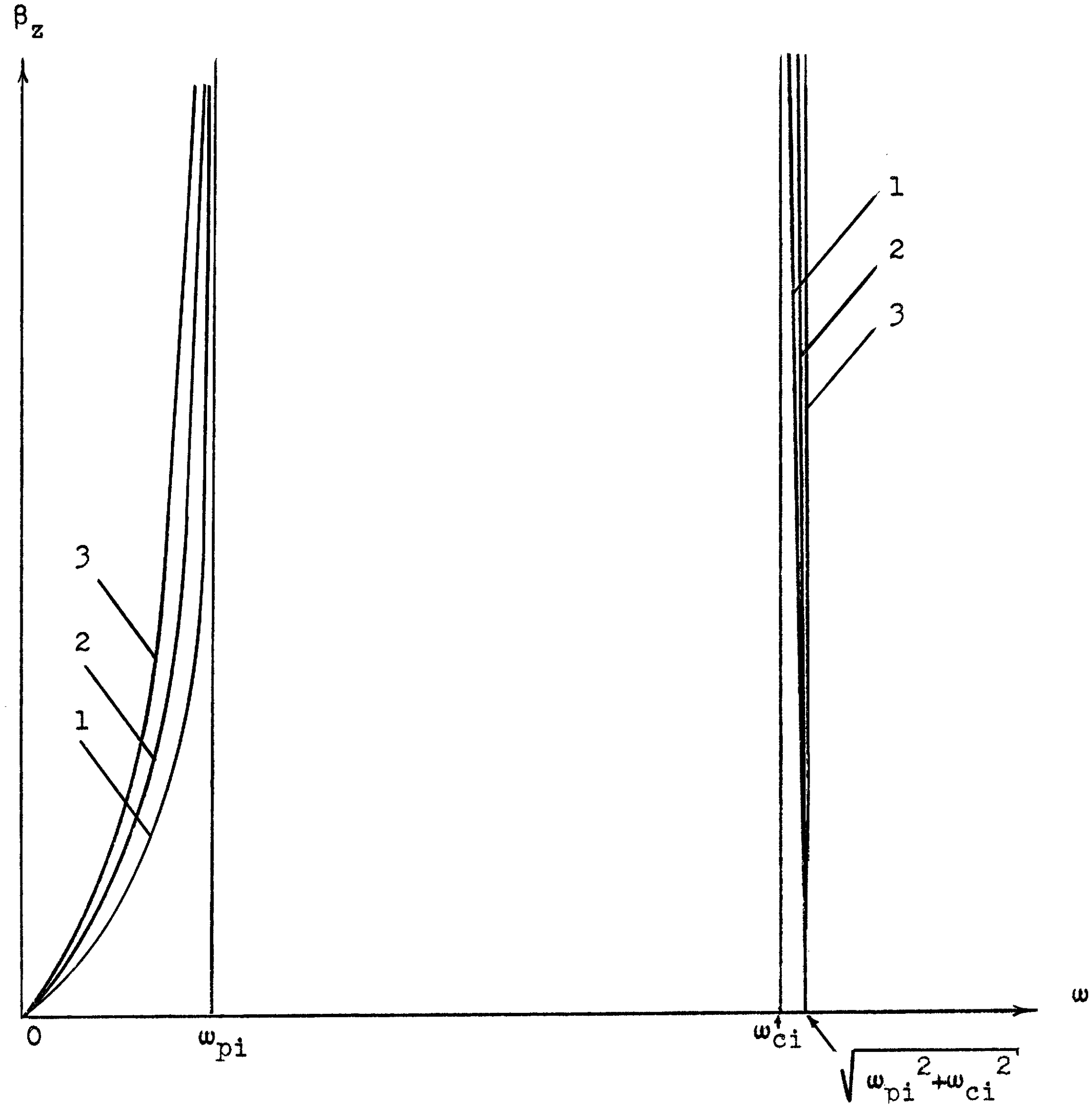
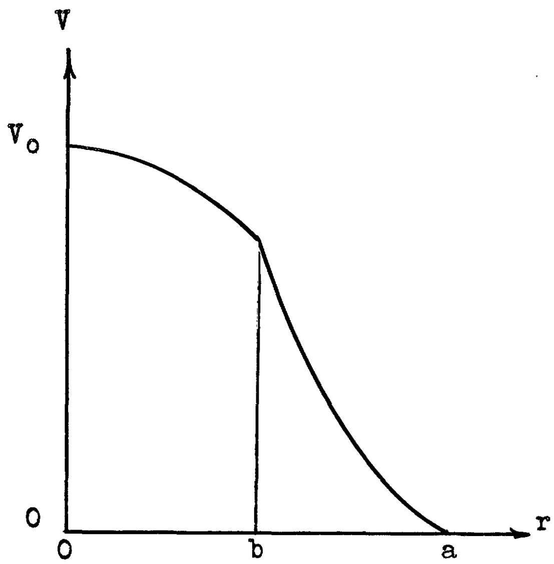
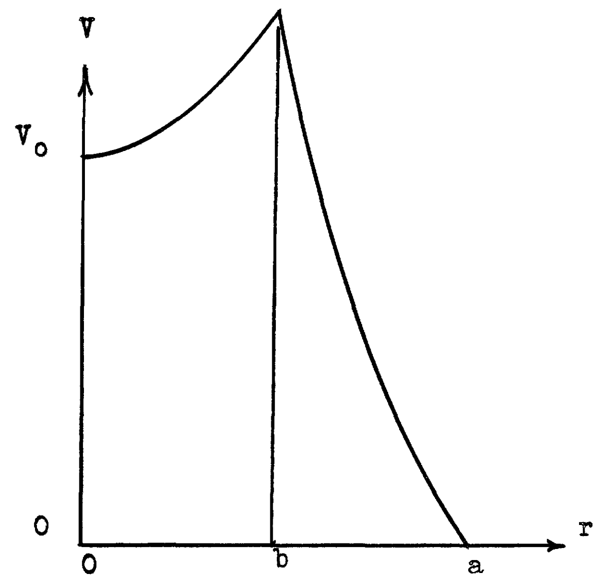


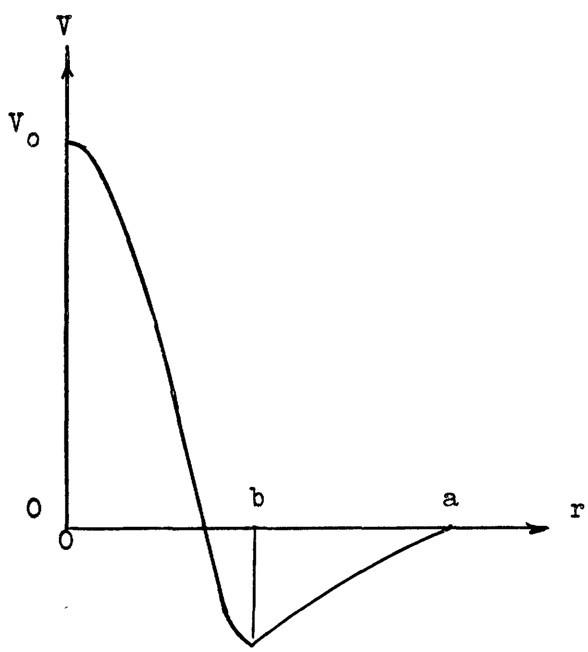
Fig.(III-12). Sketch of ω - β diagram for $b/a < 1$, $\omega_{pi} < \omega_{ci}$. The numbers indicate the order of the radial mode. All modes have the same pass bands. All waves in this case are body waves.



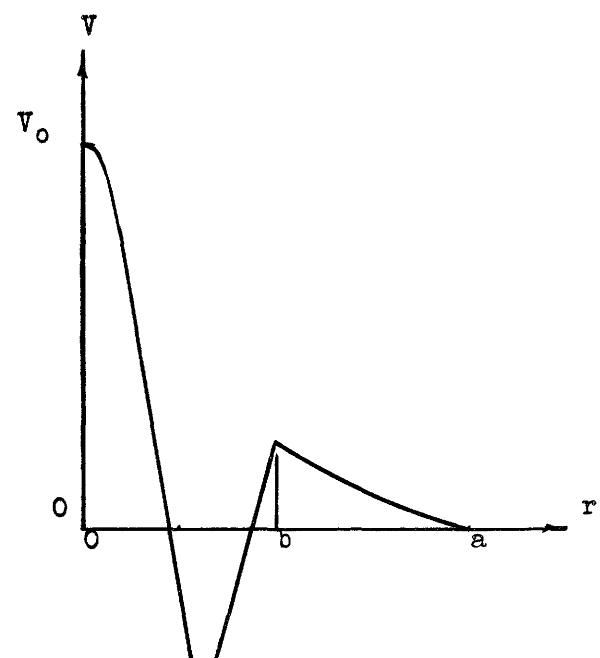
(a) Body wave, first radial mode;
$0 < \omega < \omega_{ci}$ for $\omega_{pi} > \omega_{ci}$ or
$0 < \omega < \omega_{pi}$ for $\omega_{ci} > \omega_{pi}$.



(b) Surface wave;
$\omega_{ci} < \omega < \sqrt{\frac{\omega_{ci}^2 + \omega_{pi}^2}{2}}$ for $\omega_{pi} > \omega_{ci}$.



(c) Body wave, first radial mode;
$\omega_{pi} < \omega < \sqrt{\omega_{pi}^2 + \omega_{ci}^2}$ for $\omega_{pi} > \omega_{ci}$ or
$\omega_{ci} < \omega < \sqrt{\omega_{pi}^2 + \omega_{ci}^2}$ for $\omega_{ci} > \omega_{pi}$.



(d) Body wave, second radial mode;
$0 < \omega < \omega_{ci}$ for $\omega_{pi} > \omega_{ci}$ or
$0 < \omega < \omega_{pi}$ for $\omega_{ci} > \omega_{pi}$:

Fig.(III-13) r - dependence of quasi-static potential for various conditions.

Chapter IV
A MECHANISM OF SUSTAINING OSCILLATIONS

IV - 1 Introduction

A possible mechanism of sustaining ion oscillations is studied in this chapter. The study is largely an extension of the preliminary examination of Jepsen, who calculates the energy transfer from oscillating ions to a beam for the case of one dimensional fields in the plasma[9]. Here the method of Jepsen is extended to embrace an ion plasma with the fields described in the preceeding chapter.

It is assumed that we have an ion loaded resonator consisting of a drift tube of length L and radius a, gridded on either end, through which an electron beam of radius b passes (Fig.IV-1). All the ions are assumed to be trapped in the beam and to neutralize it, and it is assumed that the field solutions of Chapter III are applicable. The energy exchange between the fields in the resonator and the electron beam is calculated.

IV - 2 Ion Loaded Resonator

If ideal grids, which are in effect shorting planes, are inserted at $z = 0$ and $z = L$ in the ion loaded cylindrical waveguide [Fig.(III-3)] we have an ion loaded cavity. The cavity resonates at an infinite number of discrete frequencies; the resonant frequencies for the circularly symmetric modes are found from the results of the previous chapter. At a low resonant frequency, ω_{mn}, the fields in the plasma in the cavity are given by the potential functions expressed in real form

$$V^I = V_o\ J_o\ (\beta_{rm}r)\ \sin\ \beta_z z\quad \cos\ \omega_{mn}t \qquad \text{(IV-2.1)}$$

or

$$V^I = V_o\ I_o\ (\alpha_{r1}r)\ \sin\ \beta_z z\quad \cos\ \omega_{1m}t \qquad \text{(IV-2.2)}$$

from Eqs.(D-6) and (III-6.14). The propagation constant β_z is now restricted so that the potential vanishes at $z = 0$ and $z = L$, that is,

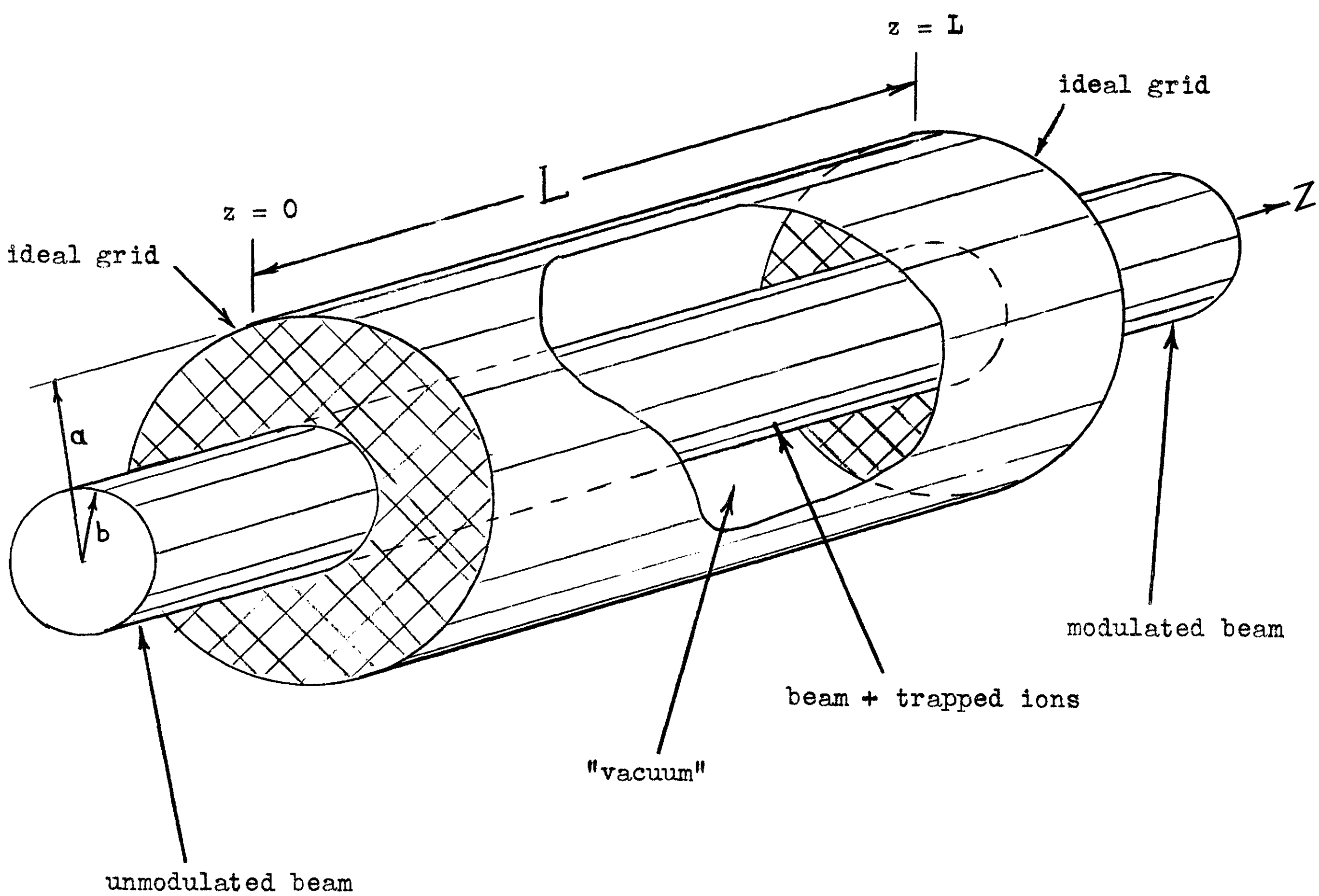


Fig.(IV-1) Ion loaded resonator traversed by a beam.

$$\beta_z L = n\pi \; ; \; n = 1, 2, \ldots \tag{IV-2.3}$$

Hence (IV-2.1) and (IV-2.2) may be rewritten

$$V^I = V_o \, J_o \, (\beta_{rm} r) \sin \frac{n\pi}{L} z \quad \cos \omega_{mn} t \tag{IV-2.4}$$

$$V^I = V_o \, I_o \, (\alpha_{r1} r) \sin \frac{n\pi}{L} z \quad \cos \omega \quad t \tag{IV-2.5}$$

The subscript n is the number of half wavelengths in the length L and the subscript m is the order of the radial mode; α_r carries the subscript 1, for, if the potential of Eq.(IV-2.5) exists at all, it will be for the first, or lowest order, radial mode.

Given an m and n, there are two low frequency resonances; for the case $\omega_{pi} > \omega_{ci}$, one occurs between 0 and $\sqrt{\frac{\omega_{pi}^2+\omega_{ci}^2}{2}}$ and the other between ω_{pi} and $\sqrt{\omega_{pi}^2+\omega_{ci}^2}$; for $\omega_{ci} > \omega_{pi}$, one occurs between 0 and ω_{pi} and the other between ω_{ci} and $\sqrt{\omega_{pi}^2+\omega_{ci}^2}$. (There is also a very high frequency resonance, but this is of no interest here since it is the same as that of the empty cavity.) The resonant frequencies are determined graphically as shown in Figs.(IV-1) and (IV-2). For a/b > 1 it is necessary to determine the resonant frequencies graphically; however for the cavity completely filled with ions, β_r is constant with frequency for a given radial mode and the resonant freauency may be found analytically from Eq.(III-4.5).

IV - 3 Energy Transfer

Jepsen calculates the energy transfer from the ion loaded resonator to an average electron of the beam for the following conditions:

1. The electrons enter the plane z = 0 with velocity $\bar{i}_z \, v_o$.
2. The transit time of the electron over the distance L, $\frac{L}{v_o} = \tau << \frac{2\pi}{\omega}$, ω being the frequency of oscillation.

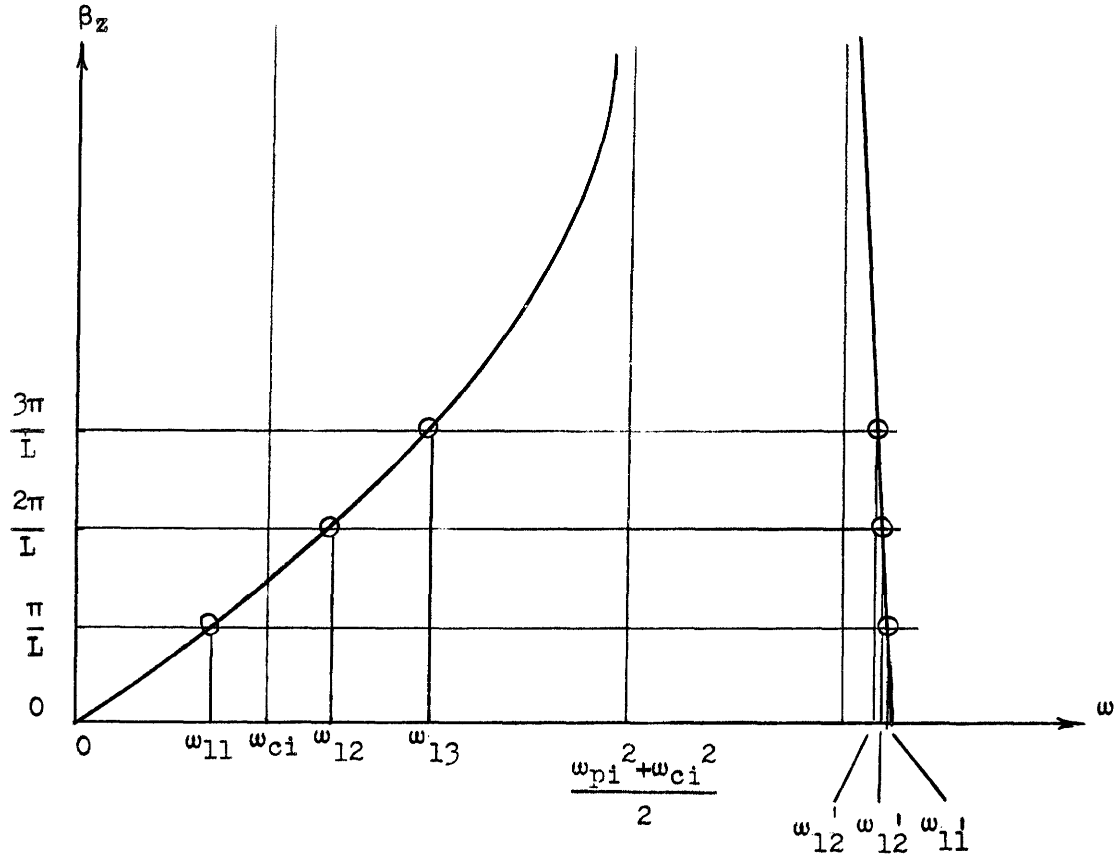


Fig. (IV-2) Graphical determination of resonant frequencies, $\omega_{pi} > \omega_{ci}$, first radial mode.

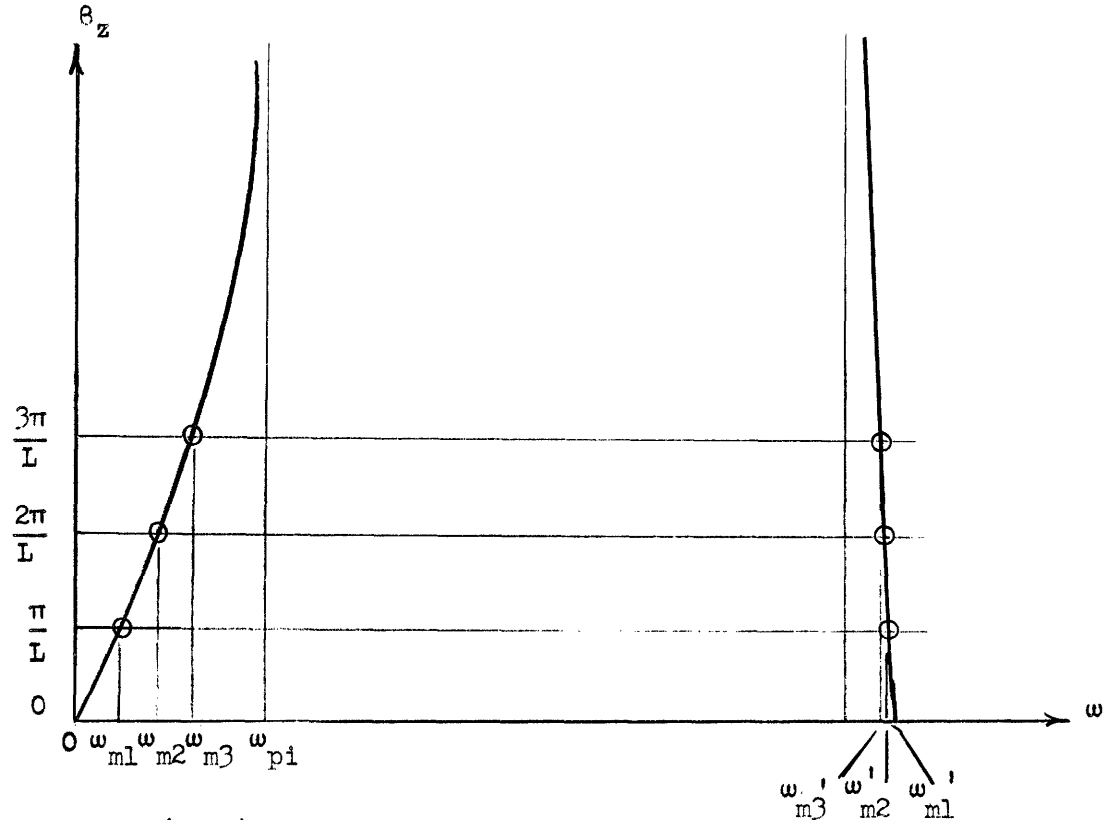


Fig. (IV-3) Graphical determination of resonant frequencies, $\omega_{ci} > \omega_{pi}$, for any given radial mode.

3. The ion resonator has an infinite cross-section so that the fields are one dimensional. There is a pure longitudinal field, E_z, meaning that the only oscillation frequency possible is $\omega = \omega_{pi}$; this is easily shown from the relations in Chapter II.

After a procedure of successive approximations, Jepsen found that on the average, the ratio between the energy transferred from the oscillating ions to an electron traversing the gap of length L and the energy of the electron at z = 0 is

$$\left.\frac{\bar{W}_e}{W_o}\right|_{z=L} = \frac{F(n\pi)}{2(n\pi)^2}\left(\frac{V_o}{U_o}\right)^2 \phi^2 \qquad \text{(IV-3.1)}$$

where V_o is the amplitude of the potential of the longitudinal field in the resonator, U_o is the dc voltage of the beam at z = 0,

$$U_o = \frac{1}{2}\ \frac{m}{e}\, v_o^2,$$

ϕ is the transit angle of the electron,

$$\phi = \frac{\omega L}{v_o} \ ,$$

and $F(x) = \left[x \sin x - \frac{1}{2}\sin^2 x - (1-\cos x)\right]$. Equation (IV-3.1) can also be interpreted as the negative of the fraction of beam power transferred to the resonator.

For n even, $F(n\pi) = 0$, and there is no energy transfer from the ions to the electrons. For n odd, $F(n\pi) = -2$, and Eq.(IV-3.1) becomes

$$\frac{\bar{W}_e}{W_o} = \frac{-1}{(n\pi)^2}\left(\frac{V_o}{U_o}\right)^2 \phi^2 \qquad \text{(IV-3.2)}$$

Here we see that for n odd, the average energy transferred from the ions to the electrons is negative. Thus the energy transfer is actually from the beam to the ions for n odd, a mechanism which could possibly supply enough energy to the ions to build up and maintain oscillations. <u>The fraction of beam power transferred to the resonator is given by $\left|\frac{\bar{W}_e}{W_o}\right|$ for n odd.</u>

The above result for the one dimensional resonator may be extended for the resonator with finite cross-section by averaging $\left|\frac{\bar{W}_e}{W_o}\right|$, which is for the finite case a function of the transverse dimensions, over the beam cross-section. Thus for the circularly symmetric modes, the time average fraction of the beam power transferred to the ions is

$$\frac{P_{ei}}{P_b} = \frac{1}{\pi b^2} \int_0^{2\pi} \int_0^b \left|\frac{\bar{W}_e}{W_o}(r)\right| r \, dr \, d\theta$$

where

$$\left|\frac{\bar{W}_e}{W_o}(r)\right| = \frac{1}{(n\pi)^2} \left(\frac{V(r)}{U_o}\right)^2 \phi^2$$

and

$$V(r) = V_o J_o(\beta_r r).$$

Hence

$$\frac{P_{ei}}{P_b} = \frac{1}{(n\pi)^2} \left(\frac{V_o}{U_o}\right)^2 \phi^2 \left[J_o^2(\beta_r b) + J_1^2(\beta_r b)\right] \quad \text{(IV-3.3)}$$

or for

$$V(r) = V_o I_o(\alpha_r r),$$

$$\frac{P_{ei}}{P_b} = \frac{1}{(n\pi)^2} \left(\frac{V_o}{U_o}\right)^2 \phi^2 \left[I_o^2(\alpha_r b) - I_1^2(\alpha_r b)\right] \quad \text{(IV-3.4)}$$

where V_o is the same as in Eqs.(IV-2.4) and (IV-2.5). The average power transferred from the beam to ions is P_{ei} and the entrance power of the beam is $P_b = I_o U_o$. Equations (IV-3.3) and (IV-3.4) differ from the one dimensional case by a geometry factor.

For the finite case, there are transverse electric fields which impart transverse velocity components to the electrons traversing the gap - this constitutes an energy flow from the resonator to the beam, the magnitude of which we now calculate. From Eq..(I-2.5) the equations for the transverse motion of an electron are

$$\frac{dv_r}{dt} = -\frac{e}{m} E_r - \omega_{ce} v_\theta \qquad \text{(IV-3.5)}$$

$$\frac{dv_\theta}{dt} = \omega_{ce} v_r \qquad \text{(IV-3.6)}$$

Since it is assumed that the transit time is very short compared with the period of the oscillation and that $v_o >> |\bar{v}_e|$, then

$$\frac{d}{dt} \approx v_o \frac{\partial}{\partial z}$$

and the equations of motion now become

$$\frac{\partial v_{er}}{\partial z} = -\frac{e}{mv_o} E_r - \beta_{ce} \; v_{e\theta} \qquad \text{(IV-3.7)}$$

$$\frac{\partial v_{e\theta}}{\partial z} = \beta_{ce} \; v_{er} \qquad \text{(IV-3.8)}$$

where $\beta_{ce} = \dfrac{\omega_{ce}}{v_o}$.

Again, since the transit time is so short, we assume that during the transit of a single electron, the field does not change with time. We further assume that the deflection of the electron is very small, so that finally the radial electric field seen by a single electron is

$$E_r = \beta_r V_o J_1 (\beta_r r_o) \sin \beta_z z \cos \omega t_o \qquad \text{(IV-3.9)}$$

or

$$E_r = -\alpha_r V_o I_1 (\alpha_r r_o) \sin \beta_z z \cos \omega t_o \qquad \text{(IV-3.10)}$$

where t_o is the time at which the electron enters the cavity and r_o is the radius at which it enters. Using Eq.(IV-3.9) in Eqs.(IV-3.7) and (IV-3.8) and the boundary conditions that at $z = 0$ $v_r = v_\theta = 0$, the solution obtained is

$$v_{er} = A (\cos \beta_z z - \cos \beta_{ce} z) \qquad \text{(IV-3.11)}$$

$$v_{e\theta} = A \left(\frac{\beta_{ce}}{\beta_z} \sin \beta_z z - \sin \beta_{ce} z\right) \qquad \text{(IV-3.12)}$$

where

$$A = \frac{e}{mv_o} \frac{\beta_z \beta_r}{\beta_z^2 - \beta_{ce}^2} V_o J_1 (\beta_r r_o) \cos \omega t_o \qquad \text{(IV-3.13)}$$

The ratio between the energy gained by an electron and the entrance energy of the electron is

$$\frac{v_{er}^2(L) + v_{e\theta}^2(L)}{v_o^2} = \frac{A^2}{v_o^2} (1 + \cos^2 n\pi - 2 \cos n\pi \cos \beta_{ce} L)$$

which for n odd becomes

$$2 \frac{A^2}{v_o^2} (1 + \cos \beta_{ce} L)$$

The average power transferred from the resonator to the beam is the value of the last expression averaged over all entrance times to, and all entrance radii r_o,

$$\frac{P_{ie}}{P_b} = \frac{\omega}{2\pi} \cdot \frac{1}{\pi b^2} \int_0^{\frac{2\pi}{\omega}} \int_0^{2\pi} \int_0^b 2 \frac{A^2}{v_o^2} (1 + \cos \beta_{ce} L) \, r_o dr_o d\theta \, dt_o$$

The evaluation of this integral yields

$$\frac{P_{ie}}{P_b} = \frac{1}{4}\left(\frac{V_o}{U_o}\right)^2\left(\frac{\beta_z\beta_r}{\beta_z^{\,2}-\beta_{ce}^{\,2}}\right)^2 \; (1 + \cos\,\beta_{ce}L)\;\Big[J_1^{\,2}(\beta_r b)$$

$$+ J_o^{\,2}(\beta_r b) - \frac{2}{\beta_r b}\,J_o\,(\beta_r b)\,J_1\,(\beta_r b)\Big] \qquad \text{(IV-3.14)}$$

which is the average power transferred from the ions to the beam in imparting transverse velocities to the electrons; note that $\beta_z = \frac{n\pi}{L}$ where n is odd.

It now remains to investigate the magnitude of the power flow from the ions to the beam P_{ie} with respect to the power flow from the beam to the ions P_{ei}. From Eqs.(IV-3.3) and (IV-3.14) we have

$$\frac{P_{ie}}{P_{ei}} = 4\,\frac{1 + \cos\,\beta_{ce}L}{\phi^2}\left(\frac{\beta_r/L}{\beta_z^{\,2}-\beta_{ce}^{\,2}}\right)^2\left[1\right.$$

$$\left. - \frac{2}{\beta_r b}\;\frac{J_o(\beta_r b)\;J_1(\beta_r b)}{J_1^{\,2}(\beta_r b) + J_o^{\,2}(\beta_r b)}\right] \qquad \text{(IV-3.15)}$$

where $\beta_z = \frac{n\pi}{L}$, n odd.

A necessary, but not a sufficient condition since losses are neglected, for sustaining oscillations is $\frac{P_{ie}}{P_{ei}} < 1$; that is the net power flow must be from the beam to the ions. Since $(1 + \cos\,\beta_{ce}L) \leq 2$ and since the quantity enclosed by the square brackets in Eq.(IV-3.15) is always between 1 and 0, we can in general conclude that for the system to oscillate,

$$\frac{P_{ie}}{P_{ei}} \leq \frac{8}{\phi^2}\left(\frac{\beta_r/L}{\beta_z^{\,2}-\beta_{ce}^{\,2}}\right)^2 \qquad \text{(IV-3.16)}$$

This expression is evaluated for a numerical example in Appendix E. Note that for the field variations of Eq.(IV-2.5) we replace $\beta_r b$ by by $j\alpha_r b$, $J_0(\beta_r b)$ by $I_0(\alpha_r b)$, and $J_1(\beta_r b)$ by $jI_1(\alpha_r b)$; gives us a similar relation for Eq.(IV-3.15) and the limiting value, Eq.(IV-3.16), is the same.

Equation (IV-3.15) indicates that there are several parameters which determine whether or not oscillations will be sustained. Presumably, the modes and frequencies of oscillation, for which the net power flow from the beam to the resonator is the greatest, build up the fastest and strongest. Thus the lower values of n seem most favorable as modes of oscillation. Also, it seems from Eq.(IV-3.15) that larger magnetic fields B_0 give larger net power flows from the beam to the ions. Modes with smaller values of β_r would be more favorable also; as β_r approaches zero, we approach the one dimensional case, where no transverse velocities are imparted to the electrons, and there is a small feedback of power from the resonator to the beam.

Because of the power losses that are incurred in physical systems through collis~~s~~ions, wall losses, ion drainage, etc., the criterion for oscillation is

$$\frac{P_{ie}}{P_{ei}} \leq 1 - \frac{P_L}{P_{ei}} \qquad \text{(IV- 3.17)}$$

where P_L is the power dissipated through losses. Thus with losses it seems that the higher order modes that could oscillate in the absence of loss would not be excited and for some cases even the lowest mode would not support oscillations if P_{ie}/P_{ei} were not small enough.

Chapter V
CONCLUDING REMARKS

The propagation of waves in a system that consists of a metal drift tube and a plasma composed of an electron beam and stationary ions was considered, and the difficulties involved in solving the total problem as we have stated it, without further assumptions and approximations, were indicated. For the particular case, in which the plasma consisted of stationary ions alone, it was found that the mathematics of the problem became simpler than that of the total problem, and field solutions and an interpretable determinantal equation were found. The problem for a plasma of ions alone was simplified by the introduction of the quasi-static assumption which allowed us to find the dispersion in an ion-loaded waveguide without too much difficulty.

A cavity that consisted of a gridded section of ion-loaded waveguide was considered. A beam was injected into this cavity; it was assumed that the fields in the cavity were unperturbed by the beam. The power transfer from the beam to the cavity P_{ei} was calculated on the basis of Jepsen's results. The power feed-back from the cavity to the beam P_{ie} was calculated on the basis that it is derived from the imparting of transverse velocity components to the electrons. From these kinematic calculations of power transfer it was concluded that the cavity traversed by the beam would be an oscillator if the right-hand side of Eq. (IV-3.15) had a value of less than unity; that is, if the net power flow was from the beam to the cavity. The criterion for oscillation is slightly modified if losses are taken into account [Eq. (IV-3.17)].

For the case of a plasma of ions alone (Chapter III), the exact field solutions, when approximated for low frequencies, indicated that the low-frequency approximations could be reached by taking a different approach - the quasi-static approach. Hence, the quasi-static assumptions, $\bar{E} = - \nabla V$, was made for the low-frequency ion solutions and valid results were obtained, the validity being exemplified by the close correlation between the exact and approximate solutions. The quasi-static approach, which has been used extensively elsewhere, offers a very simple approach to field problems that are encountered in tube study, and it is the opinion of the author that a basic proof of the validity of the quasi-static assumption is needed. Probably, with some rigor, the quasi-static assumption can be proved valid from the basic equations of Chapter I.

We must note that for some unexplained reason, the quasi-static approach gives incomplete results, as illustrated in Chapter III. Under the quasi-static assumption we obtain a determinantal equation which gives two waves, a forward-travelling and a backward-travelling wave. On the other hand, in the exact solution, there are four waves, two forward- and two backward-travelling. It so happens in the problem of a plasma of ions alone that the quasi-static approach gives us the waves that are of interest; but care must be exercised, for in different problems, some interesting modes might not show up as a result of using this approach.

The validity of the basic assumptions (uniform d.c. charge densities, no thermal motion, no collisions, etc.) and the quasi-static assumption for the case of a plasma of ions alone has been verified experimentally. Using the same assumptions, R. W. Gould and A. W. Trivelpiece have developed independently some of the theory of the ion-loaded waveguide presented in this thesis, and they have actually verified experimentally the dispersion in an ion-loaded waveguide that is predicted by the theory. The stationary ion cloud with which they worked was an electron gas derived from a mercury discharge. This work is presented in a forthcoming paper.[10]

In Chapter III, the ion-loaded waveguide was studied in two different forms - the completely filled waveguide and the partly filled waveguide. It was found for $\omega_{pi} < \omega_{ci}$ that both the filled and partly filled waveguides have the same passbands and stop bands. However, for $\omega_{ci} < \omega_{pi}$, the lowest pass bands differ for each one: for the filled waveguide, the lowest pass band is $0 < \omega < \omega_{ci}$, whereas for the partly filled waveguide, the lowest pass band is

$$0 < \omega < \sqrt{\frac{\omega_{pi}^2 + \omega_{ci}^2}{2}} .$$

For the partly filled waveguide ($b/a < 1$, $\omega_{ci} < \omega_{pi}$, lowest radial mode), the upper-frequency cutoff of the lowest pass band,

$\sqrt{\frac{\omega_{pi}^2 + \omega_{ci}^2}{2}}$, is independent of b/a; but apparently, from the mathematical results of Sections (III-5) and (III-6), as soon as we have $b/a = 1$ (the filled waveguide), the upper cutoff of the lowest radial mode becomes ω_{ci}. Such a discontinuous behaviour is not intuitively expected and, at the moment is neither mathematically nor physically understood.

The net energy transfer from the beam to the ion-loaded cavity described in Chapter IV is suggested as a possible oscillation mechanism. Whether or not this is the correct mechamism that is responsible for plasma oscillations remains to be demonstrated experimentally. One could test experimentally the criterion for oscillation [Eq.(IV-3.14), or Eq.(IV-3.17) if losses are appreciable] in order to determine the validity of this explanation.

Some further theoretical studies of propagation in plasma loaded waveguides have been made. In a forthcoming paper, to which the author has contributed, the quasi-static approach is applied to a waveguide loaded with both stationary ions and a drifting beam.[11] The results of this paper show that there are several waves in such a waveguide that resemble waves found in both travelling-wave and backward-wave devices. This paper suggests that these waves are the possible mechanism of oscillation. The differences or similarities between the mechanism described in this thesis and that of the forthcoming paper have not been investigated, but it is the feeling of the author that they are closely related.

Appendix A

DERIVATIONS OF BASIC EQUATIONS

The quantities considered are

(a) Electric Field	$0 + \bar{E}\, e^{j\omega t}$
(b) Magnetic Field	$\bar{B}_o + \mu_o\bar{H}\, e^{j\omega t}$
(c) Electron Charge Density	$-\rho_o + \rho_e\, e^{j\omega t}$
(d) Electron Current Density	$\bar{J}_o + \bar{J}_e\, e^{j\omega t}$
(e) Electron Velocity	$\bar{v}_o + \bar{v}_e\, e^{j\omega t}$
(f) Ion Charge Density	$\rho_o + \rho_i\, e^{j\omega t}$
(g) Ion Current Density	$0 + \bar{J}_i\, e^{j\omega t}$
(h) Ion Velocity	$0 + \bar{v}_i\, e^{j\omega t}$

The quantities $\bar{E}$, $\bar{H}$, ρ_e, $\bar{J}_e$, $\bar{v}_e$, ρ_i, $\bar{J}_i$, $\bar{v}_i$ are complex small signal quantities; $\bar{B}_o$, ρ_o, $\bar{J}_o$, $\bar{v}_o$ are dc quantities.

Maxwell's Equations for the small signal quantities in the plasma read

$$\nabla \times \bar{H} = \bar{J} + j\omega\epsilon_o\bar{E} \tag{A-1}$$

$$\nabla \times \bar{E} = -j\omega\mu_o\bar{H} \tag{A-2}$$

$$\nabla \cdot \bar{E} = \rho/\epsilon_o \tag{A-3}$$

$$\nabla \cdot \bar{H} = 0 \tag{A-4}$$

where $\bar{J} = \bar{J}_e + \bar{J}_i$ and $\rho = \rho_e + \rho_i$.

Now

$$\nabla \times \nabla \times \bar{E} = -j\omega\mu\, \nabla \times \bar{H} = -j\omega\mu\bar{J} + k^2\bar{E}$$

where $k^2 = \omega^2\epsilon_o\mu_o = \omega^2/c^2$. By definition of the vector Laplacian

$$\nabla^2 \bar{E} = \nabla(\nabla\cdot\bar{E}) - \nabla \times \nabla \times \bar{E}$$

and Eq. (A-3), the vector wave equation for $\bar{E}$ becomes

$$\nabla^2\bar{E} + k^2\bar{E} = j\omega\mu\bar{J} + \nabla\rho/\epsilon_o \tag{I-2.1}$$

Similarly

$$\nabla \times \nabla \times \bar{H} = \nabla \times \bar{J} + k^2\bar{H}$$

Using the definition of the vector Laplacian

$$\nabla^2\bar{H} = \nabla(\nabla\cdot\bar{H}) - \nabla \times \nabla \times \bar{H}$$

and Eq.(A-4) we arrive at the vector wave equation for $\bar{H}$

$$\nabla^2\bar{H} + k^2\bar{H} = -\nabla \times \bar{J} \qquad \text{(I-2.2)}$$

The total current density is composed of both the electron and ion current densities or

$$\bar{\mathcal{J}} = \bar{\mathcal{J}}_e + \bar{\mathcal{J}}_i = \bar{J}_o + \bar{J}\, e^{j\omega t}$$

where

$$\bar{\mathcal{J}}_e = \bar{J}_o + \bar{J}_e\, e^{j\omega t}$$

and

$$\bar{\mathcal{J}}_i = \bar{J}_i\, e^{j\omega t}$$

total ion current.

Now

$$\bar{\mathcal{J}}_e = (-\rho_o + \rho_e\, e^{j\omega t})\,(\bar{v}_o + v_e\, e^{j\omega t})$$

Hence, neglecting second order terms

$$\bar{J}_o = -\rho_o\bar{v}_o$$

and

$$\bar{J}_e = \rho_e v_o - \rho_o v_e \qquad \text{(A-5)}$$

Similarly we obtain

$$\bar{J}_i = \rho_o\bar{v}_i \qquad \text{(A-6)}$$

since the ions have no dc component of velocity. Conservation of electrons requires that

$$\nabla \cdot \bar{J}_e + j\omega\rho_e = 0 \tag{A-7}$$

the electron continuity equation, and conservation of ions,

$$\nabla \cdot \bar{J}_i + j\omega\rho_i = 0 \tag{A-8}$$

the ion continuity equation. Substituting (A-5) into (A-7) we obtain

$$\nabla \cdot (\rho_e \bar{v}_o) - \rho_o \nabla \cdot \bar{v}_e + j\omega\rho_e = 0$$

But

$$\nabla \cdot (\rho_e \bar{v}_o) = \nabla \cdot (\bar{i}_z \rho_e v_o) = \frac{\partial}{\partial z} (\rho_e v_o) = -j\beta_z v_o \rho_e$$

since it is assumed that all ac quantities have the z - dependence $e^{-j\beta_z z}$. Therefore

$$\rho_e = \frac{\nabla \cdot \bar{v}_e}{j\omega(1 - \frac{\beta_z}{\beta_e})} \rho_o \tag{A-9}$$

where $\beta_e = \frac{\omega}{v_o}$. Substitution of (A-9) into (A-5) gives

$$\bar{J}_e = \rho_o \left[\frac{\nabla \cdot \bar{v}_e}{j\omega(1 - \frac{\beta_z}{\beta_e})} \bar{v}_o - \bar{v}_e \right] \tag{A-10}$$

Finally (A-10) and (A-6) yield

$$\bar{J} = \rho_o \left[\bar{v}_i - \bar{v}_e + \frac{\nabla \cdot \bar{v}_e}{j\omega(1 - \frac{\beta_z}{\beta_e})} \bar{v}_o \right] \tag{I-2.3}$$

The total charge density is $\rho\, e^{j\omega t} = (\rho_e + \rho_i)\, e^{j\omega t}$. From equations (A-6) and (A-8) we get

$$\rho_i = - \frac{\nabla \cdot \bar{v}_i}{j\omega} \rho_o$$

Hence from equation (A-9)

$$\rho = \rho_o \left[\frac{\nabla \cdot \bar{v}_e}{j\omega(1 - \frac{\beta_z}{\beta_e})} - \frac{\nabla \cdot \bar{v}_i}{j\omega} \right] \tag{I-2.4}$$

The equation of motion of an electron is given by the non-relativistic force equation

$$m_e \frac{d\tilde{\bar{V}}_e}{dt} = - e \, (\bar{\mathcal{E}} + \tilde{\bar{V}}_e \times \bar{\mathcal{B}}) \tag{A-11}$$

where

$$\tilde{\bar{V}}_e = \bar{v}_o + \bar{v}_e \, e^{j\omega t}$$

$$\bar{\mathcal{E}} = \bar{E} \, e^{j\omega t}$$

$$\bar{\mathcal{B}} = \bar{B}_o + \mu_o \bar{H} \, e^{j\omega t}$$

Now

$$\frac{d\tilde{V}_{ex}}{dt} = \frac{\partial \tilde{V}_{ex}}{\partial t} + \frac{\partial \tilde{V}_{ex}}{\partial x}\frac{dx}{dt} + \frac{\partial \tilde{V}_{ex}}{\partial y}\frac{dy}{dt} + \frac{\partial \tilde{V}_{ex}}{\partial z}\frac{dz}{dt}$$

Since $\frac{\partial}{\partial t} = j\omega$ and $\frac{\partial}{\partial z} = -j\beta_z$, the result, when second order terms are neglected, is

$$\frac{d\tilde{V}_{ex}}{dt} = j\omega(1 - \frac{\beta_z}{\beta_e}) \, v_{ex} \, e^{j\omega t} \tag{A-12}$$

where $\beta_e = \frac{\omega}{v_o}$. Similarly

$$\frac{d\tilde{V}_{ey}}{dt} = j\omega \, (1 - \frac{\beta_z}{\beta_e}) \, v_{ey} \, e^{j\omega t} \tag{A-13}$$

and

$$\frac{d\tilde{V}_{ez}}{dt} = j\omega \, (1 - \frac{\beta_z}{\beta_e}) \, v_{ez} \, e^{j\omega t} \tag{A-14}$$

In effect, for electrons,

$$\frac{d}{dt} = j\omega \left(1 - \frac{\beta_z}{\beta_e}\right) \tag{A-15}$$

Using relations (A-12), (A-13), and (A-14) and neglecting second order terms in $\tilde{\bar{V}}_e \times \bar{\mathcal{B}}$ we find the equation of motion for the electrons is

$$\frac{-j\omega m_e}{e} \left(1 - \frac{\beta_z}{\beta_e}\right) \bar{v}_e = (\bar{E} + \bar{v}_e \times \bar{B}_o + \mu_o \bar{v}_o \times \bar{H}) \tag{I-2.5}$$

By replacing $-e$ by q, m_e by m_i, $\bar{v}_e$ by $\bar{v}_i$, and letting $v_o = 0$ since the ions are not drifting, we obtain the equation of motion for the ions,

$$\frac{j\omega m_i}{q} \bar{v}_i = (\bar{E} + \bar{v}_i \times \bar{B}_o) \tag{I-2.6}$$

Appendix B

FORMULATION OF BOUNDARY CONDITIONS

The ripple in the boundary between the plasma and the vacuum can be approximated by a sheet of ac surface charge and surface current. This is accomplished by calculating the current and charge contained in the ripple and reducing them to equivalent surface quantities on the dc boundary of the plasma [See Fig. (B-1)]. The ripple is made up of both ions and electrons; therefore, the surface charge is composed of contributions from both. The total ac surface charge density $\sigma = \sigma_e + \sigma_i$ and the total surface current density $\bar{\bar{K}} = \bar{\bar{K}}_e + \bar{\bar{K}}_i$.

For the case of a circular cylindrical plasma, the contribution of the electrons to the equivalent surface charge and current is calculated as follows. Let $\Delta b_e e^{j\omega t}$ be the radial displacement of an electron at $r = b$ due to an excitation. The disturbed radial coordinate of the electron is therefore given by $b + \Delta b_e e^{j\omega t}$ where Δb_e is a function of b, θ, and z, the z - dependence being $e^{-j\beta_z z}$; $|\Delta b_e| << b$. Consider the small element of volume on the dc boundary of the plasma as shown in figure C-1. The electron continuity equation in integral form reads $\oint\!\!\oint \bar{J}_e \cdot d\bar{A} + \frac{\partial Q_e}{\partial t} = 0$ where Q_e is the total electron charge enclosed by the volume.

$$\frac{\partial Q_e}{\partial t} = j\omega Q_e = j\omega \sigma_e e^{j\omega t}\, b\Delta\theta\Delta z$$

where σ_e is the equivalent electron surface charge density. Therefore

$$j\omega \sigma_e e^{j\omega t} + \frac{1}{b\Delta\theta\Delta z} \oint\!\!\oint \bar{J}_e \cdot d\bar{A} = 0$$

or integrating the current over the volume element, neglecting

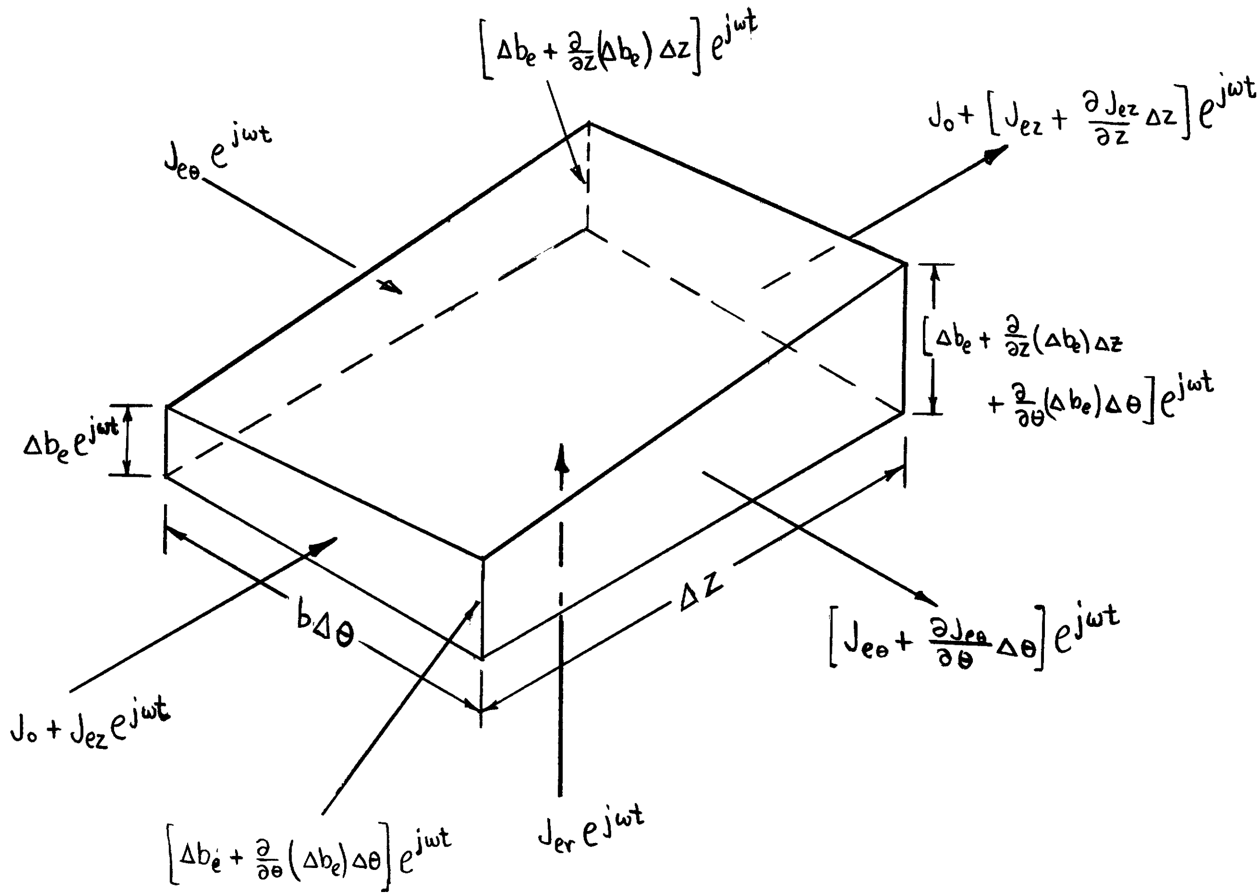


Fig. (B-1). Volume element from rippled boundary used to determine equivalent electron surface charge and surface current.

terms of the second order, and dividing out $e^{j\omega t}$, we obtain

$$j\omega \sigma_e - J_{er} + J_o \frac{\partial}{\partial z} (\Delta b_e) = 0$$

Now $J_{er} = -\rho_o v_{er}\big|_{r = b}$, $J_o = -\rho_o v_o$, and $\frac{\partial}{\partial z} = -j\beta_z$; also $v_{er} = \frac{d(\Delta b_e)}{dt} = j\omega(1 - \frac{\beta_z}{\beta_e})\Delta b_e$ from Eq.(A-15).

Consequently

$$\Delta b_e = \frac{v_{er}\big|_{r = b}}{j\omega(1 - \frac{\beta_z}{\beta_e})} \tag{B-1}$$

and

$$\sigma_e = \frac{-v_{er}\big|_{r=b}}{j\omega(1 - \frac{\beta_z}{\beta_e})} \rho_o \tag{B-2}$$

Again from Fig.(B-1), we see by neglecting second order terms, we obtain that $\bar{K}_e = \bar{i}_z K_{ez} = \bar{i}_z J_o \Delta b_e$. Therefore from Eq.(C-1)

$$\bar{K}_e = \bar{i}_z \frac{-v_{er}\big|_{r = b}}{j\beta_e(1 - \frac{\beta_z}{\beta_e})} \rho_o \tag{B-3}$$

The contributions of the ions is calculated in a similar manner with the result that

$$\Delta b_i = \frac{v_{ir}\big|_{r=b}}{j\omega} \tag{B-4}$$

$$\sigma_i = \frac{v_{ir}\big|_{r=b}}{j\omega} \rho_o \tag{B-5}$$

$$\bar{K}_i = 0 \tag{B-6}$$

Combination of the electron and ion contributions yields

$$\sigma = \rho_o \left[\frac{v_{ir}}{j\omega} - \frac{v_{er}}{j\omega(1 - \frac{\beta_z}{\beta_e})}\right]_{r = b} \qquad \text{(I-3.1)}$$

and

$$\bar{K} = \bar{i}_z \left.\frac{j\rho_o v_{er}}{\beta_e - \beta_z}\right|_{r = b} \qquad \text{(I-3.2)}$$

The procedure for calculating σ and $\bar{K}$ for a plasma of arbitrary dc cross-section follows in a similar manner with the result that

$$\sigma = \rho_o \left[\frac{\bar{v}_i \cdot \bar{n}}{j\omega} - \frac{\bar{v}_e \cdot \bar{n}}{j\omega(1 - \frac{\beta_z}{\beta_e})}\right]_{\mathcal{S}} \qquad \text{(B-7)}$$

$$\bar{K} = \bar{i}_z \left.\frac{j\rho_o \bar{v}_e \cdot \bar{n}}{\beta_e - \beta_z}\right|_{\mathcal{S}} \qquad \text{(B-8)}$$

where $\bar{n}$ is the unit vector normal to the dc surface of the plasma and pointing from the plasma to the vacuum. The velocities $\bar{v}_i$ and $\bar{v}_e$ are computed at the dc surface $\mathcal{S}$ as indicated by the subscript $\mathcal{S}$.

Appendix C

SOLUTION IN CIRCULARLY SYMMETRIC ION PLASMAS

The basic equations for this case are

$$\nabla^2 \bar{E} + k^2 \bar{E} = j\omega\mu \bar{J} + \nabla(\nabla \cdot \bar{E}) \qquad \text{(C-1)}$$

$$\bar{J} + j\frac{\omega_{ci}}{\omega} \bar{J} \times \bar{i}_z = -j\omega\epsilon_0 \frac{\omega_{pi}^2}{\omega^2} \bar{E} \qquad \text{(C-2)}$$

Equation (C-2) is obtained by combining Eqs.(III-3.2) and (III-3.3). Equation (C-2) when expressed in matrix form reads

$$\begin{bmatrix} 1 & j\frac{\omega_{ci}}{\omega} & 0 \\ -j\frac{\omega_{ci}}{\omega} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_r \\ J_\theta \\ J_z \end{bmatrix} = -j\omega\epsilon_0 \frac{\omega_{pi}^2}{\omega^2} \begin{bmatrix} E_r \\ E_\theta \\ E_z \end{bmatrix}$$

from which we obtain

$$j\omega\mu J_r = -k^2 \left[\frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} E_r - j\frac{\omega_{ci}}{\omega} \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} E_\theta \right] \qquad \text{(C-3)}$$

$$j\omega\mu J_\theta = -k^2 \left[j\frac{\omega_{ci}}{\omega} \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} E_r + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} E_\theta \right] \qquad \text{(C-4)}$$

$$j\omega\mu J_z = k^2 \frac{\omega_{pi}^2}{\omega^2} E_z \qquad \text{(C-5)}$$

Now

$$\nabla \cdot \bar{E} = \left[\bar{i}_r \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + \bar{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \bar{i}_z \frac{\partial}{\partial z} \right] \cdot \bar{E};$$

therefore, from Eqs.(III-3.4) to (III-3.6),

$$\nabla \cdot \bar{E} = (\beta_r B - j\beta_z A)\, J_0(\beta_r r)\, e^{-j\beta_z z} \qquad \text{(C-6)}$$

Also

$$\nabla(\nabla \cdot \bar{E}) = \left(\bar{i}_r \frac{\partial}{\partial r} + \bar{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \bar{i}_z \frac{\partial}{\partial z}\right)(\nabla \cdot \bar{E});$$

hence,

$$\nabla(\nabla\cdot\bar{E}) = -\bar{i}_r\beta_r(\beta_r B - j\beta_z A)\, J_1(\beta_r r)\, e^{-j\beta_z z}$$

$$-\bar{i}_z j\beta_z(\beta_r B - j\beta_z A)\, J_o(\beta_r r) e^{-j\beta_z z} \qquad \text{(C-7)}$$

The components of the vector Laplacian are

$$(\nabla^2\bar{E})_r = \frac{\partial^2 E_r}{\partial r^2} + \frac{1}{r}\frac{\partial E_r}{\partial r} - \frac{E_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 E_\theta}{\partial\theta^2} - \frac{2}{r^2}\frac{\partial E_\theta}{\partial\theta} + \frac{\partial^2 E_r}{\partial z^2}$$

$$(\nabla^2\bar{E})_\theta = \frac{\partial^2 E_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial E_\theta}{\partial r} - \frac{E_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 E_\theta}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial E_r}{\partial\theta} + \frac{\partial^2 E_\theta}{\partial z^2}$$

$$(\nabla^2\bar{E})_z = \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial\theta^2} + \frac{\partial^2 E_z}{\partial z^2}$$

from which we obtain

$$\nabla^2\bar{E} = (-\beta_r^{\,2} - \beta_z^{\,2})\,\bar{E} \qquad \text{(C-8)}$$

When Eqs.(C-3),(C-4),(C-5),(C-7), and (C-8) are substituted into Eq.(C-1) we obtain after some manipulation

$$\begin{bmatrix} -j\beta_r\beta_z & -\beta_z^{\,2}+k^2\dfrac{\omega^2-\omega_{pi}^{\,2}-\omega_{ci}^{\,2}}{\omega^2-\omega_{ci}^{\,2}} & jk^2\dfrac{\omega_c}{\omega}\dfrac{\omega_{pi}^{\,2}}{\omega^2-\omega_{ci}^{\,2}} \\ 0 & -jk^2\dfrac{\omega_{ci}}{\omega}\dfrac{\omega_{pi}^{\,2}}{\omega^2-\omega_{ci}^{\,2}} & -\beta_r^{\,2}-\beta_z^{\,2}+k^2\dfrac{\omega^2-\omega_{pi}^{\,2}-\omega_{ci}^{\,2}}{\omega^2-\omega_{ci}^{\,2}} \\ -\beta_r^{\,2}+k^2\left(1-\dfrac{\omega_{pi}^{\,2}}{\omega^2}\right) & j\beta_z\beta_r & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(C-9)}$$

Equation (C-9) is a homogeneous set, hence the determinant of the matrix is singular; the determinantal equation, Eq.(III-3.9), is the expansion of the determinant of this matrix equated to zero.

From Eq.(C-9) we obtain

$$\frac{A}{B} = \frac{-j\beta_z\beta_r}{-\beta_r^2 + k^2\left(1 - \frac{\omega_{pi}^2}{\omega^2}\right)}$$

$$\frac{C}{B} = \frac{\omega_{ci}}{\omega}\;\frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}\;\frac{jk^2}{-\beta_r^2-\beta_z^2+k^2\,\frac{\omega^2-\omega_{pi}^2-\omega_{ci}^2}{\omega^2-\omega_{ci}^2}}$$

from which we get Eqs.(III-3.7) and (III-3.8).

The ac magnetic field can now be calculated from

$$-j\omega\mu\bar{H} = \nabla \times \bar{E}$$

where

$$\nabla \times \bar{E} = \begin{vmatrix} \bar{i}_r \frac{1}{r} & \bar{i}_\theta & \bar{i}_z \frac{1}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ E_r & rE_\theta & E_z \end{vmatrix}$$

from which

$$H_r = -\frac{\beta_z}{\omega\mu}\;\frac{jk^2\frac{\omega_{ci}}{\omega}\,\frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}}{-\beta_r^2-\beta_z^2+k^2\,\frac{\omega^2-\omega_{ci}^2-\omega_{pi}^2}{\omega^2-\omega_{ci}^2}}\;BJ_1(\beta_r r)\;e^{-j\beta_z z} \quad \text{(C-10)}$$

$$H_\theta = \frac{\beta_z}{\omega\mu}\;\frac{k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)}{-\beta_r^2+k^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)}\;B\,J_1(\beta_r r)\;e^{-j\beta_z z} \quad \text{(C-11)}$$

$$H_z = -\frac{\beta_r}{\omega\mu}\ \frac{k^2\,\frac{\omega_{ci}}{\omega}\ \frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2}}{-\beta_r^2-\beta_z^2+k^2\ \frac{\omega^2-\omega_{ci}^2-\omega_{pi}^2}{\omega^2-\omega_{ci}^2}}\ B\ J_o(\beta_r r)\ e^{-j\beta_z z} \quad (C-12)$$

Appendix D

QUASI-STATIC APPROXIMATION

For low frequency solution in the ion plasma we wok from the equations

$$\bar{E} = -\nabla V \tag{D-1}$$

$$\nabla^2 V = -\rho/\epsilon_o \tag{D-2}$$

$$\bar{J} = \rho_o \bar{v} \tag{D-3}$$

$$\nabla \cdot \bar{J} + j\omega\rho = 0 \tag{D-4}$$

$$\frac{j\omega m}{q} \bar{v} = \bar{E} + \bar{v} \times B_o \tag{D-5}$$

We assume that

$$V = V_o J_o (\beta_r r) e^{-j\beta_z z} \tag{D-6}$$

Hence

$$E_r = \beta_r V_o J_1 (\beta_r r) e^{-j\beta_z z} \tag{D-7}$$

$$E_z = j\beta_z V_o J_o (\beta_r r) e^{-j\beta_z z} \tag{D-8}$$

$$E_\theta = 0 \tag{D-9}$$

When Eqs.(D-7) to (D-9) are substituted into Eqs.(C-3) to (C-5) which are derivable from (D-3) and (D-5) we obtain

$$J_r = j\omega\epsilon_o\beta_r \frac{\omega_{pi}^2}{\omega_{ci}^2-\omega^2} V_o J_1 (\beta_r r) e^{-j\beta_z z}$$

$$J_\theta = -\omega_{ci}\epsilon_o\beta_r \frac{\omega_{pi}^2}{\omega_{ci}^2-\omega^2} V_o J_1 (\beta_r r) e^{-j\beta_z z}$$

$$J_z = \omega\epsilon_o\beta_z \frac{\omega_{pi}^2}{\omega^2} V_o J_o(\beta_r r) e^{-j\beta_z z}$$

From the components of $\bar{J}$ and Eq.(D-4) we find that

$$\rho = -\epsilon_o V_o \left[\frac{\omega_{pi}^2}{\omega_{ci}^2-\omega^2} \beta_r^2 - \frac{\omega_{pi}^2}{\omega^2} \beta_z^2 \right] J_o(\beta_r r) e^{-j\beta_z z} \quad \text{(D-10)}$$

Since

$$\nabla^2 V = - \nabla \cdot \bar{E}$$

we obtain from Eqs. (D-7) to (D-9)

$$\nabla^2 V = - (\beta_r^2 + \beta_z^2) V_o J_o (\beta_r r) e^{-j\beta_z z} \quad \text{(D-11)}$$

and combining Eqs.(D-10) and (D-11) through Eq .(D-2) we get

$$-(\beta_r^2 + \beta_z^2) = \frac{\omega_{pi}^2}{\omega_{ci}^2-\omega^2} \beta_r^2 - \frac{\omega_{pi}^2}{\omega^2} \beta_z^2$$

which gives

$$\beta_z^2 = \frac{\omega^2(\omega_{pi}^2+\omega_{ci}^2-\omega^2)}{(\omega_{pi}^2-\omega^2)\ (\omega_{ci}^2-\omega^2)} \beta_r^2 \quad \text{(D-12)}$$

From the above steps we see that the assumed potential function V is consistent with the revised set of basic equations, Eqs.(D-1) to (D-5), and that the final results, Eqs.(D-7) to (D-9) and (D-12), are the same as Eqs.(III-4.5), (III-4.7) to (III-4.9) which were obtained as a result of using the exact set of small signal equations.

The θ - component of magnetic field is now computed from

$$\oint \bar{H} \cdot d\bar{s} = \int_A (\bar{J} + j\omega\epsilon\bar{E}) \cdot d\bar{A}$$

Hence

$$2\pi r H_\theta = \int_0^r (J_z + j\omega\epsilon E_z)\, 2\pi r dr$$

or

$$H_\theta = \frac{1}{r}\, \omega\epsilon\beta_z \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) V_o e^{-j\beta z} \int_0^r r\, J_o(\beta_r r)\, dr$$

from which

$$H_\theta = -\omega\epsilon \frac{\beta_z}{\beta_r} \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) V_o J_1(\beta_r r)\, e^{-j\beta_z z} \qquad \text{(D-13)}$$

Recalling that $V_o = B/\beta_r$ we see that Eq.(D-13), the quasi-static θ - component of magnetic field, is identical to the exact component, Eq.(C-11), when k^2 is neglected in comparison with $\beta_r^{\ 2}$.

Appendix E

A NUMERICAL COMPUTATION OF LIMITING VALUE OF P_{ie}/P_{ei}.

From Eq.(IV-3.15)

$$\frac{P_{ie}}{P_{ei}} \leq \frac{8}{\phi^2} \left(\frac{\beta_r/L}{\beta_z^2-\beta_{ce}^2}\right)^2$$

Let

$$\omega = 6 \times 10^6 \text{ sec}^{-1}$$

$$L = 40 \text{ cm}$$

$$v_o = 2 \times 10^9 \text{ cm sec}^{-1}$$

$$\beta_r = 2.5/.25 \text{ cm} = 10 \text{ cm}^{-1}$$

$$n = 1$$

$$B_o = 500 \text{ Gauss}$$

Hence

$$\phi = \frac{6 \times 10^6 \text{ sec}^{-1} \times 40 \text{ cm}}{2 \times 10^9 \text{ cm sec}^{-1}} = .120$$

$$\beta_r/L = 10 \text{ cm}^{-1}/40 \text{ cm} = .250 \text{ cm}^{-2}$$

$$\beta_z = 1 \times \pi/40 \text{ cm} = .0785 \text{ cm}^{-1}$$

$$\beta_{ce} = \frac{\omega_{ce}}{v_o} = \frac{2\pi \times 2.8 \times 10^6 \text{ sec}^{-1} \text{ Gauss}^{-1} \times 500 \text{ Gauss}}{2 \times 10^9 \text{ cm sec}^{-1}} = 4.40 \text{ cm}^{-1}$$

Therefore

$$\frac{P_{ie}}{P_{ei}} \leq .09$$

and we see that for this example the power flow from the beam to the cavity greatly exceeds the power flow the cavity to the beam thus indicating that sustained oscillations are possible for this case.

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